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PERMUTABILITY OF INFERENCES AND CATEGORICAL EQUIVALENCE OF DERIVATIONS IN IMLL

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Основным результатом статьи является доказательсво теоремы, что отношение эквивалентности на выводах интуиционистской мультипликативной линейной логики, порожденное перестановками правил в стиле Клини, совпадает с отношением эквивалентности, порожденным структурой свободной симметрической моноидальной замкнутой категории на том же исчислении. Как известно, в категорной структуре формулы играют роль объектов, связкам соответствуют функторы, а морфизмами являются классы эквивалентности выводов секвенций вида $A \to B$. Интерес к этому исчислению объясняется, с одной стороны, богатством категорных моделей (от модулей над коммутативными кольцами с 1 до множеств с отмеченной точкой и частичных порядков), а с другой – его нетривиальностью с точки зрения теории доказательств. Основной результат статьи позволит улучшить сложностные характеристики существующих разрешающих алоритмов для категорной эквивалентности. Ряд лемм сформулирован для более сильных отношений эквивалентности, соотвествующих несвободным категорным моделям, что позволяет применить методы теории доказательств для их изучения, в том числе, в задаче проверки коммутативности диаграмм в категориях, преставляющих интерес с точки зрения алгоритмов, используеых в системах символьных вычислений.

We show that two cut-free derivations in intuitionistic multiplicative linear logic with unit are equivalent as morphisms of a free symmetric monoidal closed category iff they are inter-permutable in the sense of Kleene. We use this result to propose a new deciding algorithm for categorical equivalence of morphisms and to determine the size of some homsets, correcting a close claim by Szabo.

Introduction. According to folk wisdom, two cut-free derivations in a sequent calculus are to be considered "the same" iff one can be obtained from the other using permutations of inferences as described by Kleene in [8]. In [3], Dyckoff and Pinto give a precise justification to this statement for Gentzen's LJ system, by showing that two cut-free derivations are inter-permutable iff they have the same image under Prawitz's mapping from LJ to NJ. In this paper, we give a new justification to the above statement for intuitionistic multiplicative linear logic with tensor unit (from now on IMLL), by relating inter-permutability of cut-free derivations to categorical equivalence of morphisms in free symmetric monoidal closed categories (from now on SMCC's).

Since the pioneering work by Lambek, it is widely understood that, given a logic, one can form a category whose objects are formulas and whose morphisms *equivalence classes* of proofs (see [10]). In this setting, the logic's connectives may be regarded as functors, the logic's inference rules as natural transformations and the logical system itself as a category freely generated by a set of atoms. Being freely generated, this category can be used to derive general information about all categories of the same structure. The most famous example of this idea is given by the relations between intuitionistic logic and cartesian closed categories (the equivalence relation on derivations there coincides with the notion provided by Prawitz' normalization.) The study of fragments of linear logic in connection with SMCC's and other types of closed categories is also relatively well-developed (see [1]). The question of the proper equivalence on proofs is quite delicate and of particular relevance to linear logicians, since there is no generally accepted notion of normal form for derivations in linear logic.

The main result of this paper is that two cut-free derivations in IMLL are equivalent as morphisms of a free SMCC iff they are inter-permutable. Thus, a precise justification to the folklore inter-permutability statement for IMLL is given. The proof includes, though, consideration of stronger equivalence relations and opens the door for future study of non-free categories using the techniques of permutation of rules. The paper suggests also a notion of (non-unique) normal form of sequential derivations in IMLL and proposes a new deciding algorithm for their equivalence based on this notion, as well as some new calculations of the number of non-equivalent derivations.

To place properly this paper in the context of research in linear logic and its categorical semantics, it should be recalled that several decision procedures for categorical equivalence of proofs were suggested: the procedures based on lambda-terms and normalization ([12]), on generalization of proof-nets ([17, 2]), algebraic methods ([18]), Gentzen-style methods ([9, 4, 15]). This paper answers natural questions important for future development of methods based on Gentzen-style approach, in particular the results concerning permutability will permit to further improve the algorithms of low polynomial complexity suggested in ([15]).

1. IMLL. The cut-free sequent calculus we're considering is built over a set of atoms $\mathcal{A}$. Formulas are recursively defined by $A = a \mid I \mid A \otimes A \mid A \multimap A$, where a stands for any atom of $\mathcal{A}$, and the inference system is defined by the following axiom and rules.

$$\frac{}{A \vdash A}$$

$$\frac{A, B, \Gamma \vdash C}{A \otimes B, \Gamma \vdash C}\otimes_{\mathcal{L}} \qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\otimes_{\mathcal{R}}$$

$$\frac{\Gamma \vdash A \quad B, \Delta \vdash C}{\Gamma, A \multimap B, \Delta \vdash C}\multimap_{\mathcal{L}} \qquad \frac{A, \Gamma \vdash B}{\Gamma \vdash A \multimap B}\multimap_{\mathcal{R}}$$

$$\frac{\Gamma \vdash I \quad \Delta \vdash A}{\Gamma, \Delta \vdash A}wkn$$

The formula introduced by an inference rule is called its *main* formula: $A \otimes B$ (resp. $A \multimap B$) is the main formula of of $\otimes_{\mathcal{L}}$ and $\otimes_{\mathcal{R}}$ (resp. of $\multimap_{\mathcal{L}}$ and $\multimap_{\mathcal{R}}$). The formulas used to construct the main formula of an inference rule are called its *side* formulas: A and B are the side formulas of $\otimes_{\mathcal{L}}$, $\otimes_{\mathcal{R}}$, $\multimap_{\mathcal{L}}$ and $\multimap_{\mathcal{R}}$.

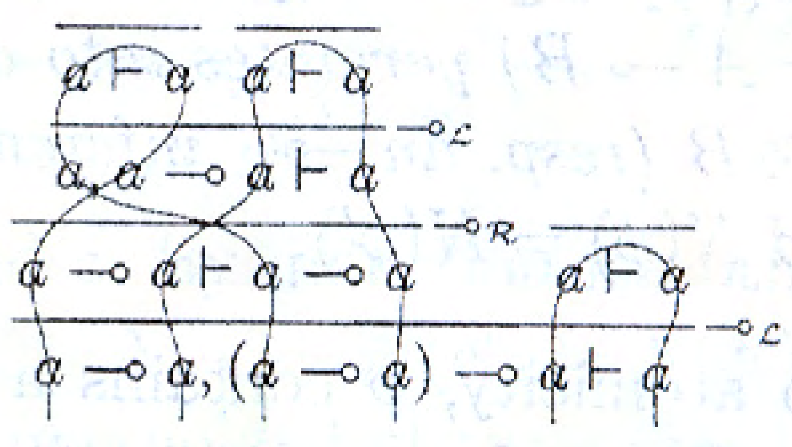
We will consider premises both as sequents and derivations. When it is not clear from the context, they will be called sequent-premises and derivation-premises respectively.

Atomic derivations and balanced sequents. There is no hope to state the inter-permutability of, say, the axiom $A \otimes B \vdash A \otimes B$ and the derivation

$$\dfrac{\dfrac{A \vdash A \quad B \vdash B}{A, B \vdash A \otimes B}\otimes_{\mathcal{R}}}{A \otimes B \vdash A \otimes B}\otimes_{\mathcal{L}} .$$

Thus, when talking about inter-permutability of derivations, we're considering *atomic* derivations only, that is derivations such that axioms are of the form $a \vdash a$ or $I \vdash I$. This is to be related to the necessity of introducing η-conversions when comparing normal forms of natural-deduction derivations. However, this is not a strong constraint since every axiom may replaced by its atomic derivation.

The axioms $a \vdash a$ in an atomic derivation ϕ are in one-to-one correspondence with (disjoint) pairs of occurrences of the atom a in the end-sequent of ϕ. As an example, consider



The variables of different pairs may be given different names. The end-sequent obtained that way is called *balanced* (in the example above, we obtain $a \multimap b, (a \multimap b) \multimap c \vdash c$). Every atom in it occurs exactly twice and with opposite variances. We shall consider balanced sequents only.

2. Inter-permutability of derivations. 2.1. Permutations of inferences. The following table presents the permutations of inferences we're considering. The intersection of a row (r) and a column (c) refers to the permutability of (r) and (c) in a derivation where (r) lies immediately above (c). An empty cell indicates that the permutation is forbidden, ✓ that it is possible with no restriction, 1 and 2 that it is possible under some condition:

1 the main formula of (r) is not a side formula of (c);

2 the side formulas of (c) do not belong to different premises of (r).

	$\otimes_{\mathcal{L}}$	$\otimes_{\mathcal{R}}$	$\multimap_{\mathcal{L}}$	$\multimap_{\mathcal{R}}$	*wkn*
$\otimes_{\mathcal{L}}$	1	✓	1	1	✓
$\otimes_{\mathcal{R}}$	2		1		✓
$\multimap_{\mathcal{L}}$	2	✓	1	2	✓
$\multimap_{\mathcal{R}}$	✓		1		✓
wkn	2	✓	✓	2	✓

Two derivations ϕ and ϕ' of the same sequent are said to be *inter-permutable* if one can be obtained from the other using permutations of inferences as described above. Equivalently, we'll say that ϕ permutes into ϕ', resp. that ϕ' permutes into ϕ.

D e f i n i t i o n 1. (Minimal derivation, normal derivation, normal set) *We call a derivation* minimal *if it does not permute into a derivation ending by* wkn, $\otimes_{\mathcal{L}}$ *or* $\multimap_{\mathcal{L}}$. *We call a derivation* ϕ normal *if every* left *derivation-premise of* $\multimap_{\mathcal{L}}$ *in* ϕ *is minimal. Let* $N(\phi)$ *be the set of all left sequent-premises of* $\multimap_{\mathcal{L}}$ *in a normal derivation* ϕ. *We call* $N(\phi)$ *the* normal set *of* ϕ.

L e m m a 1. *Every subderivation of a normal derivation is normal.*

Let ϕ be a normal derivation. If ϕ is an axiom, then $N(\phi)$ is empty. If ϕ is ending by $\otimes_{\mathcal{L}}$ or $\multimap_{\mathcal{R}}$ with premise γ, then $N(\phi) = N(\gamma)$. If ϕ if ending by *wkn* or $\otimes_{\mathcal{R}}$ with premises γ and δ then $N(\phi) = N(\gamma) \cup N(\delta)$. Finally, if ϕ is ending by $\multimap_{\mathcal{L}}$ with left and right premises two derivations γ and δ of the sequents $\Gamma \vdash A$ and $B, \Delta \vdash C$ respectively, then γ is minimal and $N(\phi) = N(\gamma) \cup \{\Gamma \vdash A\} \cup N(\delta)$. Since we're considering balanced sequents only, the unions above are partitions.

L e m m a 2. (Derivations ending by $\otimes_{\mathcal{L}}$ or $\multimap_{\mathcal{R}}$) *Every derivation* ϕ *of a sequent* $A \otimes B, \Gamma \vdash C$ *(resp. of a sequent* $\Gamma \vdash A \multimap B$*) permutes into a derivation* ϕ' *ending by an* $\otimes_{\mathcal{L}}$ *inference with main formula* $A \otimes B$ *(resp. an* $\multimap_{\mathcal{R}}$ *inference with main formula* $A \multimap B$*). If* ϕ *is normal, then so is* ϕ' *and* $N(\phi) = N(\phi')$.

P r o o f. ($\otimes_{\mathcal{L}}$ case.) Due to atomicity, ϕ contains a subderivation ending by a $\otimes_{\mathcal{L}}$ inference with main formula $A \otimes B$, which we write $A \otimes_{\mathcal{L}} B$. We proceed by induction on the height of $A \otimes_{\mathcal{L}} B$ in ϕ.
Base. If ϕ is ending by $A \otimes_{\mathcal{L}} B$, then take ϕ' to be ϕ. If ϕ is normal, then trivially so is ϕ' and $N(\phi) = N(\phi')$.
Induction. If ϕ is ending by a unary inference (r) with premise a derivation γ, then by induction hypothesis γ permutes into a derivation γ' ending by $A \otimes_{\mathcal{L}} B$. Whatever (r), as the table of permutations shows, $A \otimes_{\mathcal{L}} B$ may be pushed below (r). Take ϕ' to be the derivation obtained this way. If ϕ is normal, then so is γ. By induction hypothesis, so is γ' then so is ϕ' and $N(\phi) = N(\gamma) = N(\gamma') = N(\phi')$. If ϕ is ending by a binary inference, then $A \otimes B$ belongs to one of its premises and the previous argument applies in a similar way. The proof for the $\multimap_{\mathcal{R}}$ case uses a similar induction. ■

L e m m a 3. (Derivations ending by *wkn* or $\otimes_{\mathcal{R}}$) *If a sequent* S *can be presented as* $\Gamma, \Delta \vdash B$ *(resp.* $\Gamma, \Delta \vdash A \otimes B$*) in such a way that the sequents* $T = \Gamma \vdash I$ *(resp.* $T = \Gamma \vdash A$*) and* $U = \Delta \vdash B$ *are balanced, then every derivation* ϕ *of* S *permutes into a derivation* ϕ' *ending by* wkn *(resp.* $\otimes_{\mathcal{R}}$*) with premises* T *and* U. *If* ϕ *is normal, then so is* ϕ' *and* $N(\phi) = N(\phi')$.

P r o o f. (*wkn* case.) We proceed by induction on the height of ϕ.
Base. If ϕ is an axiom, then the stated implication is trivially true.
Induction. If ϕ is ending by $\otimes_{\mathcal{L}}$ with premise a derivation γ and main formula $C \otimes D$, say, Γ may be presented as $\Gamma', C \otimes D$. Since $\Gamma', C \otimes D \vdash I$ is balanced, so is $\Gamma', C, D \vdash I$. By induction hypothesis γ permutes into a derivation γ' ending by *wkn* with premises $\Gamma', C, D \vdash I$ and $\Delta \vdash B$. As the table of permutations shows, *wkn* may be pushed below $\otimes_{\mathcal{L}}$. Take ϕ' to be the derivation obtained this way. If ϕ is normal, then so is γ. By induction hypothesis, so is γ' then so is ϕ' and $N(\phi) = N(\gamma) = N(\gamma') = N(\phi')$. If ϕ is ending by $\multimap_{\mathcal{R}}$ with main formula $C \multimap D$, then $C, \Delta \vdash D$ is balanced and the previous argument applies.

If ϕ is ending by an inference (r) which is *wkn* or $\otimes_{\mathcal{R}}$ with premises some derivations γ and δ, then let $\Gamma' \vdash C$ and $\Delta' \vdash D$ be the end-sequents of γ and δ respectively (either $C = I$ and $D = B$ or $C \otimes D = B$). Now $\Gamma, \Delta = \Gamma', \Delta'$ so Γ' and Δ' may be written as $\Gamma' \cap \Gamma, \Gamma' \cap \Delta$ and $\Delta' \cap \Gamma, \Delta' \cap \Delta$. Then $\Gamma' \cap \Gamma$ and $\Delta' \cap \Gamma$ are necessarily balanced. One of these intersections may be empty. To be as general as possible, we assume that both are non empty. Then by induction hypothesis, γ and δ eventually into derivations γ' and δ' ending by *wkn* with left premises $\Gamma' \cap \Gamma \vdash I$ and $\Delta' \cap \Gamma \vdash I$ respectively. That is, ϕ permutes into

$$\dfrac{\dfrac{\overline{\Gamma' \cap \Gamma \vdash I}\,\gamma'_L \quad \overline{\Gamma' \cap \Delta \vdash C}\,\gamma'_R}{\Gamma' \vdash C}\,wkn \quad \dfrac{\overline{\Delta' \cap \Gamma \vdash I}\,\delta'_L \quad \overline{\Delta' \cap \Delta \vdash D}\,\delta'_R}{\Delta' \vdash D}\,wkn}{\Gamma', \Delta' \vdash B}\,(r) .$$

As the table of permutations allows us to, we take ϕ' to be the derivation

$$\dfrac{\dfrac{\overline{\Gamma' \cap \Gamma \vdash I}\,\gamma'_L \quad \overline{\Delta' \cap \Gamma \vdash I}\,\delta'_L}{\Gamma \vdash I}\,wkn \quad \dfrac{\overline{\Gamma' \cap \Delta \vdash C}\,\gamma'_R \quad \overline{\Delta' \cap \Delta \vdash D}\,\delta'_R}{\Delta \vdash B}\,(r)}{\Gamma, \Delta \vdash B}\,wkn .$$

If ϕ is normal, then so are γ and δ. By induction hypothesis, so are γ' and δ' then so is ϕ'. Finally, $N(\phi) = N(\gamma) \cup N(\delta) = N(\gamma') \cup N(\delta') = N(\phi')$. If ϕ is ending by $\multimap_{\mathcal{L}}$, then an analysis similar to the previous one holds but two cases have to be considered: the main formula belongs either to Γ or to Δ.
The proof for the $\otimes_{\mathcal{R}}$ case uses a similar induction. ■

2.2. Stronger equivalences. If this section, we relate non inter-permutability to "twisted" left introductions implication. Let ϕ and ϕ' be two derivations ending by $\multimap_{\mathcal{L}}$ with different main formulas

$$\dfrac{\overline{\Gamma, [A' \multimap B'] \vdash A}\,\gamma \quad \overline{B, [A' \multimap B'], \Delta \vdash C}\,\delta}{\Gamma, A \multimap B, A' \multimap B', \Delta \vdash C}\,\multimap_{\mathcal{L}} ;$$

$$\dfrac{\overline{\Gamma', [A \multimap B] \vdash A'}\,\gamma' \quad \overline{B', [A \multimap B], \Delta' \vdash C}\,\delta'}{\Gamma', A \multimap B, A' \multimap B', \Delta' \vdash C}\,\multimap_{\mathcal{L}} ;$$

where $A' \multimap B'$, resp. $A \multimap B$ belongs either to the left or to the right premise of $\multimap_{\mathcal{L}}$ in ϕ, resp. ϕ'. There are four cases to consider.

RR If $A' \multimap B'$ and $A \multimap B$ belong to the right premises in ϕ and ϕ' respectively, then $\Gamma \cap \Gamma' \vdash I$ is balanced. If it is not empty, then up to permutation ϕ and ϕ' are ending by the same *wkn* (lemma 3).

RL If $A' \multimap B'$ belongs to the right premise in ϕ and $A \multimap B$ belongs to the left premise in ϕ', then $\Gamma \cap \Delta' \vdash I$ is balanced. As previously, if it is not empty then up to permutation ϕ and ϕ' are ending by the same *wkn*.

LR This case may be considered the same as the previous one.

LL If $A' \multimap B'$ and $A \multimap B$ belong to the left premises in ϕ and ϕ' respectively, then $\Delta \cap \Delta' \vdash C$ is balanced. Once again, if it is not empty then up to permutation ϕ and ϕ' are ending by the same *wkn*.

Substitutions of I. Let ϕ be a derivation of some sequent S. Let α substitute I for some of atoms in S. Then $\phi\star\alpha$ stands for the derivation obtained from ϕ by "erasing" the formulas of S that have become isomorphic to I due to α. The way $\phi\star\alpha$ is obtained from ϕ doesn't matter here (see [15]), the action of the $\star$ operation is made clear by the lemmas below.

From now on, $\sim$ stands for an equivalence stable by permutations of inferences and substitutions of I.

L e m m a 4. *(RR case.) Let ϕ and ϕ' be the derivations*

$$\frac{\dfrac{}{\Gamma\vdash A}\gamma \quad \dfrac{}{B,\Gamma',A'\multimap B',\Theta\vdash C}\delta}{\Gamma,A\multimap B,\Gamma',A'\multimap B',\Theta\vdash C}\multimap_{\mathcal{L}}\ ,\qquad \frac{\dfrac{}{\Gamma'\vdash A'}\gamma' \quad \dfrac{}{B',\Gamma,A\multimap B,\Theta\vdash C}\delta'}{\Gamma',A'\multimap B',\Gamma,A\multimap B,\Theta\vdash C}\multimap_{\mathcal{L}}\ .$$

Let α (resp. α') substitute I for every atom in $\Gamma\vdash A$ (resp. $\Gamma'\vdash A'$). Then ϕ and ϕ' are $\sim$-equivalent iff δ and δ' are $\sim$-equivalent to

$$\frac{\dfrac{}{\Gamma'\vdash A'}\gamma' \quad \dfrac{}{B',B,\Theta\vdash C}\delta\star\alpha}{\Gamma',A'\multimap B',B,\Theta\vdash C}\multimap_{\mathcal{L}} \quad \text{and} \quad \frac{\dfrac{}{\Gamma\vdash A}\gamma \quad \dfrac{}{B,B',\Theta\vdash C}\delta\star\alpha'}{\Gamma,A\multimap B,B',\Theta\vdash C}\multimap_{\mathcal{L}}$$

respectively, that is to $\phi'\star\alpha$ and $\phi\star\alpha'$ respectively. Notice that if ϕ and ϕ' are not inter-permutable, then δ and $\phi'\star\alpha$ or *δ' and $\phi\star\alpha'$ aren't either.*

P r o o f. Assume that $\phi\sim\phi'$. Then $\delta=\phi\star\alpha\sim\phi'\star\alpha$ and $\delta'=\phi'\star\alpha'\sim\phi\star\alpha'$. Assume that $\delta\sim\phi'\star\alpha$ and $\delta'\sim\phi\star\alpha'$. Then $\delta\star\alpha'\sim(\phi'\star\alpha)\star\alpha'=(\phi'\star\alpha')\star\alpha\sim\delta'\star\alpha$. Now, as the table of permutations shows, the $\multimap_{\mathcal{L}}$ inferences in ϕ (resp. in ϕ') may be permuted with each other. So $\phi\sim\phi'$. ■

L e m m a 5. *(RL case.) Let ϕ and ϕ' be the derivations*

$$\frac{\dfrac{}{\Gamma\vdash A}\gamma \quad \dfrac{}{B,\Theta,A'\multimap B',\Delta'\vdash C}\delta}{\Gamma,A\multimap B,\Theta,A'\multimap B',\Delta'\vdash C}\multimap_{\mathcal{L}}\ ,\qquad \frac{\dfrac{}{\Gamma,A\multimap B,\Theta\vdash A'}\gamma' \quad \dfrac{}{B',\Delta'\vdash C}\delta'}{\Gamma,A\multimap B,\Theta,A'\multimap B',\Delta'\vdash C}\multimap_{\mathcal{L}}\ .$$

Let α substitute I for every atom in $\Gamma\vdash A$. Then ϕ and ϕ' are $\sim$-equivalent iff δ and γ' are $\sim$-equivalent to

$$\phi'\star\alpha=\frac{\dfrac{}{B,\Theta\vdash A'}\gamma'\star\alpha \quad \dfrac{}{B',\Delta'\vdash C}\delta'}{B,\Theta,A'\multimap B',\Delta'\vdash C}\multimap_{\mathcal{L}} \quad \text{and}\ \gamma''=\frac{\dfrac{}{\Gamma\vdash A}\gamma \quad \dfrac{}{B,\Theta\vdash A'}\gamma'\star\alpha}{\Gamma,A\multimap B,\Theta\vdash A'}\multimap_{\mathcal{L}}$$

respectively. Notice that if ϕ and ϕ' are not inter-permutable, then δ and $\phi'\star\alpha$ or *γ' and γ'' aren't either.*

P r o o f. Assume that $\phi\sim\phi'$. Then $\delta=\phi\star\alpha\sim\phi'\star\alpha$ and as the table of permutations shows, the $\multimap_{\mathcal{L}}$ inferences in ϕ may be permuted with each other. So ϕ and ϕ' are $\equiv$-equivalent to derivations ending by $\multimap_{\mathcal{L}}$ with the same premises, and $\gamma'\sim\gamma''$. Now, assume that $\delta\sim\phi'\star\alpha$ and $\gamma'\sim\gamma''$. Once again, as the table of permutations shows, the $\multimap_{\mathcal{L}}$ inferences in ϕ (resp. in ϕ') may be permuted with each other. So $\phi\sim\phi'$. ■

Finally, the LL case.

D e f i n i t i o n 2. (Critical pair) *Let ϕ and ϕ' be the derivations*

$$\frac{\dfrac{}{\Theta,A'\multimap B',\Delta'\vdash A}\gamma \quad \dfrac{}{B,\Delta\vdash I}\delta}{\Theta,A'\multimap B',\Delta',A\multimap B,\Delta\vdash I}\multimap_{\mathcal{L}}\ ,\qquad \frac{\dfrac{}{\Theta,A\multimap B,\Delta\vdash A'}\gamma' \quad \dfrac{}{B',\Delta'\vdash I}\delta'}{\Theta,A\multimap B,\Delta,A'\multimap B',\Delta'\vdash I}\multimap_{\mathcal{L}}\ .$$

The pair $\langle\phi,\phi'\rangle$ is called critical *if γ and γ' are minimal.*

The definition of critical pair captures the problem of "twisted" left introductions of implication noticed by Voreadou in [18].

E x a m p l e 1. *Let ϕ and ϕ' be the derivations*

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{\overline{A, A \multimap I \vdash I}}}{A \multimap I \vdash A \multimap I} \quad \overline{I \vdash I}}{A \multimap I, (A \multimap I) \multimap I \vdash I}}{(A \multimap I) \multimap I \vdash (A \multimap I) \multimap I} \quad \overline{I \vdash I}}{(A \multimap I) \multimap I, ((A \multimap I) \multimap I) \multimap I \vdash I} \quad , \quad \dfrac{\dfrac{\dfrac{\overline{\overline{A, A \multimap I \vdash I}}}{A \vdash (A \multimap I) \multimap I} \quad \overline{I \vdash I}}{A, ((A \multimap I) \multimap I) \multimap I \vdash I}}{((A \multimap I) \multimap I) \multimap I \vdash A \multimap I} \quad \overline{I \vdash I}}{((A \multimap I) \multimap I) \multimap I, (A \multimap I) \multimap I \vdash I} \quad .$$

The pair $\langle \phi, \phi' \rangle$ is obviously critical. It has to be related to so-called "triple unit problem" mentioned by Kelly and Mac Lane in [7]. We give a detailed analysis of a generalization of this problem in section .

P r o p o s i t i o n 6. *Let ϕ and ϕ' be two $\sim$-equivalent derivations of the same sequent. If ϕ and ϕ' are not inter-permutable, then a critical pair $\langle \psi, \psi' \rangle$ can be extracted from $\langle \phi, \phi' \rangle$ by permutations of inferences and substitutions of I, such that ψ and ψ' are $\sim$-equivalent.*

P r o o f. By induction on the height of the derivations.
Base. If ϕ and ϕ' are the same axiom, then the stated implication is trivially true.
Induction. If ϕ permutes into a derivation ending by $\otimes_{\mathcal{L}}$ or $\multimap_{\mathcal{R}}$ with premise a derivation γ, then ϕ' permutes into a derivation ending by $\otimes_{\mathcal{L}}$ or $\multimap_{\mathcal{R}}$ with premise a derivation γ' of the same sequent as γ (lemma 2), and $\gamma \sim \gamma'$. But γ and γ' are not inter-permutable (indeed, if they were inter-permutable, then so would be ϕ and ϕ'). By induction hypothesis, the stated implication is true. The same way, if ϕ permutes into a derivation ending by *wkn* or $\otimes_{\mathcal{R}}$ with premises some derivations γ and δ, then ϕ' permutes into a derivation ending by *wkn* or $\otimes_{\mathcal{R}}$ with premises some derivation γ' and δ' of the same sequents as γ and δ respectively (lemma 3), $\gamma \sim \gamma'$ and $\delta \sim \delta'$. But γ and γ' *or* δ and δ' are not inter-permutable (indeed, if both γ and γ' *and* δ and δ' were inter-permutable, then so would be ϕ and ϕ'). By induction hypothesis, the stated implication is true.
Finally, assume that ϕ and ϕ' are both ending by $\multimap_{\mathcal{L}}$. Due to permutability of inferences, we may assume that the left premises are minimal. If these premises are derivations of the same sequent, then the same argument as for *wkn* and $\otimes_{\mathcal{R}}$ applies. Else, the RR, RL and LL cases stated above have to be considered. If ϕ and ϕ' are as in lemma 4 (RR case), then $\delta \sim \phi' \star \alpha$ and $\delta' \sim \phi \star \alpha'$ but δ and $\phi' \star \alpha$ *or* δ' and $\phi \star \alpha'$ are not inter-permutable (indeed, if both δ and $\phi' \star \alpha$ *and* δ' and $\phi \star \alpha'$ were inter-permutable, then so would be ϕ and ϕ'). By induction hypothesis, the stated implication is true. A similar argument applies to the RL case (see lemma 5). Now, if ϕ and ϕ' are in the LL case, then $\langle \phi, \phi' \rangle$ is critical and by hypothesis, $\phi \sim \phi'$. ■

2.3. Composition of derivations. We mentioned already that we consider cut-free derivations only. This includes derivations obtained by *elimination* of cuts. Let ϕ and ψ be some derivations of $\Gamma \vdash A$ and $A, \Delta \vdash B$ respectively. Then $\psi \circ \phi$ stands for the *composite* derivation of ϕ and ψ, that is the derivation obtained by cut-elimination from

$$\dfrac{\overline{\overline{\Gamma \vdash A}}^{\,\phi} \quad \overline{\overline{A, \Delta \vdash B}}^{\,\psi}}{\Gamma, \Delta \vdash B}\,cut\ .$$

The process of cut-elimination for IMLL is described in [7, 6]. As a particular case, let ϕ and ψ be the derivations

$$\dfrac{\overline{A,\Delta\vdash B}^{\,\delta}}{\Delta\vdash A\multimap B}\multimap_{\mathcal{R}} \quad\text{and}\quad \dfrac{\overline{\Gamma\vdash A}^{\,\gamma}\quad \overline{B,\Theta\vdash C}^{\,\theta}}{\Gamma\vdash A\multimap B,\Theta\vdash C}\multimap_{\mathcal{L}}.$$

Then $\psi\circ\phi$ stands for the derivation obtained by cut-elimination from

$$\dfrac{\overline{\Gamma\vdash A}^{\,\gamma}\quad \dfrac{\overline{A,\Delta\vdash B}^{\,\delta}\quad \overline{B,\Theta\vdash C}^{\,\theta}}{A,\Delta,\Theta\vdash C}cut}{\Gamma,\Delta,\Theta\vdash C}cut\;,$$

that is $(\theta\circ\delta)\circ\gamma$.

Lemma 7. *Let ϕ and θ be the derivations*

$$\dfrac{\dfrac{\overline{A,\Gamma\vdash B}^{\,\gamma}\quad \overline{C,\Delta\vdash D}^{\,\delta}}{A,\Gamma,B\multimap C,\Delta\vdash D}\multimap_{\mathcal{L}}}{\Gamma,B\multimap C,\Delta\vdash A\multimap D}\multimap_{\mathcal{R}} \quad\text{and}\quad \overline{A\multimap D,\Theta\vdash E}^{\,\theta}.$$

If $\theta\circ\phi$ is minimal, then so is θ.

Proof. By contraposition.

- If θ permutes into a derivation ending by *wkn*, $\otimes_{\mathcal{L}}$ or $\multimap_{\mathcal{L}}$ with main formula in Θ, then so does $\theta\circ\phi$.
- If θ permutes into a derivation ending by $\multimap_{\mathcal{L}}$ with main formula $A\multimap D$, then $\theta\circ\phi$ permutes into a derivation ending by $\multimap_{\mathcal{L}}$ with main formula $B\multimap C$.

■

From now on, $\sim$ stands for an equivalence stable by permutations of inferences, substitutions of I and composition.

2.4. Reduced critical pairs.

Definition 3. (Reduced critical pair) *Let ϕ and ϕ' be the derivations*

$$\dfrac{\overline{\Theta,A'\multimap I\vdash A}^{\,\gamma}\quad \overline{I\vdash I}}{\Theta,A'\multimap I,A\multimap I\vdash I}\multimap_{\mathcal{L}}\;,\quad \dfrac{\overline{\Theta,A\multimap I\vdash A'}^{\,\gamma'}\quad \overline{I\vdash I}}{\Theta,A\multimap I,A'\multimap I\vdash I}\multimap_{\mathcal{L}}.$$

If $\langle\phi,\phi'\rangle$ is critical, then it is called reduced.

The difference between reduced critical pairs and general critical pairs (definition 2) is that w.r.t. definition 2, both δ and δ' are the axiom $I\vdash I$. The purpose of this section is to prove the following lemma.

Lemma 8. *Let ϕ and ϕ' be two $\sim$-equivalent derivations of the same sequent. If ϕ and ϕ' are not inter-permutable, then a* reduced *critical pair $\langle\psi,\psi'\rangle$ can be extracted from $\langle\phi,\phi'\rangle$ by permutations of inferences, substitutions of I and composition, such that ψ and ψ' are $\sim$-equivalent.*

Without loss of generality, we can assume that $\langle \phi, \phi' \rangle$ is critical (proposition 6). Let ϕ and ϕ' be the derivations

$$\dfrac{\overline{\overline{\Theta, A' \multimap B', \Delta' \vdash A}}^{\gamma} \quad \overline{\overline{B, \Delta \vdash I}}^{\delta}}{\Theta, A' \multimap B', \Delta', A \multimap B, \Delta \vdash I} \multimap_{\mathcal{L}} \text{ and } \dfrac{\overline{\overline{\Theta, A \multimap B, \Delta \vdash A'}}^{\gamma'} \quad \overline{\overline{B', \Delta' \vdash I}}^{\delta'}}{\Theta, A \multimap B, \Delta, A' \multimap B', \Delta' \vdash I} \multimap_{\mathcal{L}} .$$

Let α substitute I for all atoms in $B, \Delta \vdash I$ and ε be the derivation

$$\dfrac{\dfrac{\overline{\overline{A \vdash A}} \quad \overline{\overline{B, \Delta \vdash I}}^{\delta}}{A, A \multimap B, \Delta \vdash I} \multimap_{\mathcal{L}}}{A \multimap B, \Delta \vdash A \multimap I} \multimap_{\mathcal{R}} .$$

Then $\phi \sim (\phi \star \alpha) \circ \varepsilon$ and $\phi' \sim (\phi' \star \alpha) \circ \varepsilon$ iff $\gamma' \sim (\gamma' \star \alpha) \circ \varepsilon$. Assume that $\langle \phi, \phi' \rangle$ is critical and $\phi \sim \phi'$. Then γ' is minimal, $\phi' \sim \phi \sim (\phi \star \alpha) \circ \varepsilon \sim (\phi' \star \alpha) \circ \varepsilon$ and $\gamma' \sim (\gamma' \star \alpha) \circ \varepsilon$.

- If γ' permutes into $(\gamma' \star \alpha) \circ \varepsilon$, then $\gamma' \star \alpha$ is minimal (lemma 7); $\langle \phi \star \alpha, \phi' \star \alpha \rangle$ is critical and $\phi \star \alpha \sim \phi' \star \alpha$.
- Else, a critical pair $\langle \psi, \psi' \rangle$ can be extracted from $\langle \gamma', (\gamma' \star \alpha) \circ \varepsilon \rangle$ by permutations of inferences and substitutions of I, such that ψ and ψ' are $\sim$-equivalent (proposition 6).

Using this argument, one may prove by induction on the length of the end-sequent of ϕ and ϕ' that both δ and δ' can be assumed to be the axiom $I \vdash I$.

2.5. $\sim$-critical pairs.

D e f i n i t i o n 4. ($\sim$-critical pair) *Let ϕ and ϕ' be the derivations*

$$\dfrac{\overline{\overline{\Theta, A' \multimap I \vdash A}}^{\gamma} \quad \overline{I \vdash I}}{\Theta, A' \multimap I, A \multimap I \vdash I} \multimap_{\mathcal{L}} , \quad \dfrac{\overline{\overline{\Theta, A \multimap I \vdash A'}}^{\gamma'} \quad \overline{I \vdash I}}{\Theta, A \multimap I, A' \multimap I \vdash I} \multimap_{\mathcal{L}} .$$

The pair $\langle \phi, \phi' \rangle$ is called $\sim$-critical *if γ and γ' are* $\sim$-minimal, *that is if γ and γ' are not $\sim$-equivalent to derivations ending by* wkn, $\otimes_{\mathcal{L}}$ *or* $\multimap_{\mathcal{L}}$.

The difference between $\sim$-critical pairs and reduced critical pairs is that γ and γ' are minimal not for permutability but for $\sim$. Thus, a reduced critical is not $\sim$-critical only if γ or γ' is $\sim$-equivalent to a derivation ending by $\multimap_{\mathcal{L}}$. The purpose of this section is to prove the following proposition.

P r o p o s i t i o n 9. *Let ϕ and ϕ' be two $\sim$-equivalent derivations of the same sequent. If ϕ and ϕ' are not inter-permutable, then there exists a $\sim$-critical pair $\langle \psi, \psi' \rangle$ such that ψ and ψ' are $\sim$-equivalent.*

Without loss of generality, we can assume that $\langle \phi, \phi' \rangle$ is a reduced critical pair (lemma 8). Let ϕ and ϕ' be the derivations

$$\dfrac{\overline{\overline{\Theta, A' \multimap I \vdash A}}^{\gamma} \quad \overline{I \vdash I}}{\Theta, A' \multimap I, A \multimap I \vdash I} \multimap_{\mathcal{L}} , \quad \dfrac{\overline{\overline{\Theta, A \multimap I \vdash A'}}^{\gamma'} \quad \overline{I \vdash I}}{\Theta, A \multimap I, A' \multimap I \vdash I} \multimap_{\mathcal{L}} .$$

Assume that $\langle \phi, \phi' \rangle$ is critical and $\phi \sim \phi'$.

- If both γ and γ' are $\sim$-minimal, then $\langle\phi,\phi'\rangle$ is $\sim$-critical and $\phi \sim \phi'$.
- Else, assume that say, γ' is $\sim$-equivalent to a derivation γ'' ending by $\multimap_{\mathcal{L}}$. Then a reduced critical pair $\langle\psi,\psi'\rangle$ can be extracted from $\langle\gamma',\gamma''\rangle$ by permutations of inferences, substitutions of I and composition, such that ψ and ψ' are $\sim$-equivalent (lemme 8).

R e m a r k 1. *Let's mention that in this case, $\langle\psi,\psi'\rangle$ is not extracted from $\langle\phi,\phi'\rangle$ since its construction relies on γ'', which we don't know. However, the end-sequent of ψ and ψ' is extracted from the end-sequent of ϕ and ϕ'.*

Using the previous argument, one may prove proposition 9 by induction on the length of the end-sequent of ϕ and ϕ'.

3. Main result. The categorical equivalence $\equiv$ on derivations of IMLL over a set of atoms $\mathcal{A}$ is introduced via a translation of the derivations of a sequent $\Gamma \vdash A$ into the arrows $[\Gamma] \to A$ of the free SMCC generated by $\mathcal{A}$, where $[\Gamma]$ stands for the "tensor" product of all propositions in Γ. Two derivations are defined to be $\equiv$-equivalent if their images in the free SMCC generated by $\mathcal{A}$ are equivalent, where the equivalence on arrows is defined to be the smallest equivalence relation generated by the axioms for categories, functoriality of $\otimes$ and $\multimap$, naturality of basic transformations, and axioms specific to SMCC's. We won't describe neither the translation nor the equivalence on arrows since they are fully detailed in Soloviev's [14]. What matters here is that $\equiv$ is stable by permutations of inferences, substitutions of I and composition (and such that a derivation with cuts is $\equiv$-equivalent to the derivation obtained by removing cuts from it), as shown in [15].

T h e o r e m 10. *Two derivations of the same sequent are $\equiv$-equivalent iff they are inter-permutable.*

The right-to-left direction is quite obvious. A derivation ϕ is $\equiv$-equivalent to itself and $\equiv$ is stable by permutations of inferences. Then if ϕ permutes into a derivation ϕ', they are $\equiv$-equivalent. The other direction is less immediate but a partial result is already known, concerning 2-sequents. A sequent $\Gamma \vdash A$ is called a 2-sequent is A contains at most one connective and if every formula in Γ contains at most two connectives. Soloviev proved the following [14, corollary 14. 6].

P r o p o s i t i o n 11. *If $\langle\phi,\phi'\rangle$ is a $\equiv$-critical pair then ϕ and ϕ' are not $\equiv$-equivalent.*

Thus, for the case of 2-sequents we may use reduction *ad absurdum*. Let ϕ and ϕ' be two $\equiv$-equivalent derivations of the same 2-sequent. Assume that they are not inter-permutable. Since $\equiv$ is stable by permutations of inferences, substitutions of I and composition, there exists a $\equiv$-critical pair $\langle\psi,\psi'\rangle$ of derivations of a 2-sequent such that ψ and ψ' are $\equiv$-equivalent (proposition 9 and remark 1). We get a contradiction.

Now, we reduce the general case to the case of 2-sequents in the following way. From every derivation ϕ, we construct a derivation ψ of a 2-sequent. This construction is such that given two derivations ϕ and ϕ' of the same sequent, ψ and ψ' are two derivations of the same 2-sequent and

- if ϕ and ϕ' are not inter-permutable then ψ and ψ' are not either;
- if ϕ and ϕ' are $\equiv$-equivalent, then so are ψ and ψ'.

To emphasize the duality aspect of the construction, we introduce polarized substitutions. For every proposition A, let $\mathcal{N}(A)$ and $\mathcal{P}(A)$ denote the sets of negative and positive occurrences of atoms in A respectively. Let s and t be two substitutions. The application of the *polarized substitution* $\sigma = \langle s, t\rangle$ to a proposition A is defined by the simultaneous application of s and t to the occurrences of $\mathcal{N}(A)$ and $\mathcal{P}(A)$ respectively. To make the definition precise, we introduce the *dual* $\bar{\sigma} = \langle t, s\rangle$ of σ. Then $I\sigma = I\bar{\sigma} = I$; for every atom a, $a\sigma = at$ and $a\bar{\sigma} = as$; for every propositions A and B

$$(A \otimes B)\sigma = A\sigma \otimes B\sigma \text{ and } (A \otimes B)\bar{\sigma} = A\bar{\sigma} \otimes B\bar{\sigma}, \tag{1}$$
$$(A \multimap B)\sigma = A\bar{\sigma} \multimap B\sigma \text{ and } (A \multimap B)\bar{\sigma} = A\sigma \multimap B\bar{\sigma}. \tag{2}$$

Notice that for every proposition A, $A\bar{\bar{\sigma}} = A\sigma$. Now, let Π_A be the set

$$\{a\bar{\sigma} \multimap a \mid a \in \mathcal{N}(A), a\bar{\sigma} \neq a\} \cup \{a \multimap a\sigma \mid a \in \mathcal{P}(A), a \neq a\sigma\}. \tag{3}$$

As previously, to make the definition precise, we introduce the *dual* $\bar{\Pi}$ of Π. Then $\Pi_I = \bar{\Pi}_I = \varnothing$; for every atom a, $\Pi_a = \{a \multimap a\sigma\}$ if $a \neq a\sigma$, $\varnothing$ otherwise and $\bar{\Pi}_a = \{a\bar{\sigma} \multimap a\}$ if $a\bar{\sigma} \neq a$, $\varnothing$ otherwise; for every propositions A and B

$$\Pi_{A\otimes B} = \Pi_A \cup \Pi_B \text{ and } \bar{\Pi}_{A\otimes B} = \bar{\Pi}_A \cup \bar{\Pi}_B, \tag{4}$$
$$\Pi_{A\multimap B} = \bar{\Pi}_A \cup \Pi_B \text{ and } \bar{\Pi}_{A\multimap B} = \Pi_A \cup \bar{\Pi}_B. \tag{5}$$

Notice that for every proposition A, $\bar{\bar{\Pi}}_A = \Pi_A$.

For every derivations ϕ and ϕ', we are now able to construct the derivations ψ and ψ' mentioned above: we see every formula that contains "too much" connectives as obtained by application of a polarized substitution. As an example, we see $A \multimap B$ as $(a \multimap b)\sigma$ for some polarized substitution σ such that $a\bar{\sigma} = A$ and $b\sigma = B$. Then, for every derivation ϕ of $A \multimap B, \Gamma \vdash X$, that is $(a \multimap b)\sigma, \Gamma \vdash X$, we construct a derivation ψ of $a \multimap b, A \multimap a, b \multimap B, \Gamma \vdash X$, that is $a \multimap b, \Pi_{a\multimap b}, \Gamma \vdash X$ (the atoms a and b have to be *fresh* for ϕ so that the end-sequent of ψ is balanced). If $A \multimap a$ or $b \multimap B$ still contains "too much" connectives, we proceed the same way.

P r o p o s i t i o n 12. *Let σ be a polarized substitution and A a proposition. For every derivation ϕ of $S = A\sigma, \Gamma \vdash X$, resp. of $S = \Gamma \vdash A\bar{\sigma}$, there exists a derivation $\phi \circ \pi_A$ of $A, \Pi_A, \Gamma \vdash X$, resp. a derivation $\bar{\pi}_A \circ \phi$ of $\Gamma, \bar{\Pi}_A \vdash A$. For every derivations ϕ and ϕ' of S, if ϕ and ϕ' are inter-permutable, then so are $\phi \circ \pi_A$ and $\phi' \circ \pi_A$, resp. $\bar{\pi}_A \circ \phi$ and $\bar{\pi}_A \circ \phi'$.*

R e m a r k 2. *The notations $\phi \circ \pi_A$ and $\bar{\pi}_A \circ \phi$ may evoke composition. This is not innocent. In a system with* cut, *these derivations may be obtain from ϕ and some derivation π_A and $\bar{\pi}_A$ of $A, \Pi_A \vdash A\sigma$ and $A\bar{\sigma}, \bar{\Pi}_A \vdash A$ respectively. It may be proved by induction on A that these derivations exist. Now, it should be clear from this remark and the proof below that, if two derivations ϕ and ϕ' of the same sequent are $\equiv$-equivalent, then so are $\phi \circ \pi_A$ and $\phi' \circ \pi_A$, resp. $\bar{\pi}_A \circ \phi$ and $\bar{\pi}_A \circ \phi'$.*

P r o o f. By structural induction on A. We prove the first point and let the reader (easily) check the second one, having in mind that at each step of the induction, since ϕ and ϕ' are inter-permutable, they can be presented the same way.

Base. Let x be the constant I or an atom such that $x = x\sigma$, resp. $x\bar{\sigma} = x$. Let ϕ be a derivation of $x, \Gamma \vdash X$, resp. $\Gamma \vdash x$. Take $\phi \circ \pi_x$, resp. $\bar{\pi}_x \circ \phi$ to be ϕ. Let a be an atom such that $a \neq a\sigma$, resp. $a\bar{\sigma} \neq a$. Let ϕ be a derivation of $a\sigma, \Gamma \vdash X$, resp. $\Gamma \vdash a\bar{\sigma}$. Take $\phi \circ \pi_a$, resp. $\bar{\pi}_a \circ \phi$ to be

$$\frac{\overline{a \vdash a} \quad \overline{\overline{a\sigma, \Gamma \vdash X}}^{\phi}}{a, a \multimap a\sigma, \Gamma \vdash X}\multimap_{\mathcal{L}} \text{, resp. } \frac{\overline{\overline{\Gamma \vdash a\bar{\sigma}}}^{\phi} \quad \overline{a \vdash a}}{\Gamma, a\bar{\sigma} \multimap a \vdash a}\multimap_{\mathcal{L}} .$$

Induction. Let A and B be two propositions. Then, let ϕ be a derivation of $(A \otimes B)\sigma, \Gamma \vdash X$, resp. $\Gamma \vdash (A \otimes B)\bar{\sigma}$. We may assume that ϕ permutes into

$$\frac{\overline{\overline{A\sigma, B\sigma, \Gamma \vdash X}}^{\gamma}}{A\sigma \otimes B\sigma, \Gamma \vdash X}\otimes_{\mathcal{L}} \text{, resp. } \frac{\overline{\overline{\Delta \vdash A\bar{\sigma}}}^{\delta} \quad \overline{\overline{\Theta \vdash B\bar{\sigma}}}^{\theta}}{\Delta, \Theta \vdash A\bar{\sigma} \otimes B\bar{\sigma}}\otimes_{\mathcal{R}} .$$

Take $\phi \circ \pi_{A \otimes B}$, resp. $\bar{\pi}_{A \otimes B} \circ \phi$ to be

$$\frac{\overline{\overline{A, \Pi_A, B, \Pi_B, \Gamma \vdash X}}^{(\gamma \circ \pi_B) \circ \pi_A}}{A \otimes B, \Pi_A, \Pi_B, \Gamma \vdash X}\otimes_{\mathcal{L}} \text{, resp. } \frac{\overline{\overline{\Delta, \bar{\Pi}_A \vdash A}}^{\bar{\pi}_A \circ \delta} \quad \overline{\overline{\Theta, \bar{\Pi}_B \vdash B}}^{\bar{\pi}_B \circ \theta}}{\Delta, \bar{\Pi}_A, \Theta, \bar{\Pi}_B \vdash A \otimes B}\otimes_{\mathcal{R}} .$$

Let ϕ be a derivation of $(A \multimap B)\sigma, \Gamma \vdash X$, resp. $\Gamma \vdash (A \multimap B)\bar{\sigma}$. We may assume that ϕ permutes into

$$\frac{\overline{\overline{\Delta \vdash A\bar{\sigma}}}^{\delta} \quad \overline{\overline{B\sigma, \Theta \vdash X}}^{\theta}}{\Delta, A\bar{\sigma} \multimap B\sigma, \Theta \vdash X}\multimap_{\mathcal{L}} \text{, resp. } \frac{\overline{\overline{A\sigma, \Gamma \vdash B\bar{\sigma}}}^{\gamma}}{\Gamma \vdash A\sigma \multimap B\bar{\sigma}}\multimap_{\mathcal{R}} .$$

Take $\phi \circ \pi_{A \multimap B}$, resp. $\bar{\pi}_{A \multimap B} \circ \phi$ to be

$$\frac{\overline{\overline{\Delta, \bar{\Pi}_A \vdash A}}^{\bar{\pi}_A \circ \delta} \quad \overline{\overline{B, \Pi_B, \Theta \vdash X}}^{\theta \circ \pi_B}}{\Delta, \bar{\Pi}_A, A \multimap B, \Pi_B, \Theta \vdash X}\multimap_{\mathcal{L}} \text{, resp. } \frac{\overline{\overline{A, \Pi_A, \Gamma, \bar{\Pi}_B \vdash B}}^{\bar{\pi}_B \circ (\gamma \circ \pi_A)}}{\Pi_A, \Gamma, \bar{\Pi}_B \vdash A \multimap B}\multimap_{\mathcal{R}} .$$

■

Now, we want to show that if ϕ and ϕ' are not inter-permutable, then $\phi \circ \pi_A$ and $\phi' \circ \pi_A$, resp. $\bar{\pi}_A \circ \phi$ and $\bar{\pi}_A \circ \phi'$ aren't either. Actually, we're going to show that ϕ and ϕ' are inter-permutable *iff* $\phi \circ \pi_A$ and $\phi' \circ \pi_A$, resp. $\bar{\pi}_A \circ \phi$ and $\bar{\pi}_A \circ \phi'$ are inter-permutable.

P r o p o s i t i o n 13. *Let σ be a polarized substitution and A a proposition. For every derivation ϕ of $S = A, \Pi_A, \Gamma \vdash X$, resp. of $S = \Gamma, \bar{\Pi}_A \vdash A$, there exists a derivation $\phi \bullet \pi_A$ of $A\sigma, \Gamma \vdash X$, resp. a derivation $\bar{\pi}_A \bullet \phi$ of $\Gamma \vdash A\bar{\sigma}$. For every derivations ϕ and ϕ' of S, if ϕ and ϕ' are inter-permutable, then so are $\phi \bullet \pi_A$ and $\phi' \bullet \pi_A$, resp. $\bar{\pi}_A \bullet \phi$ and $\bar{\pi}_A \bullet \phi'$.*

P r o o f. Similar to the previous one, by structural induction on A.
Base. Let x be the constant I or an atom such that $x = x\sigma$, resp. $x\bar{\sigma} = x$. Let ϕ be a derivation of $x, \Gamma \vdash X$, resp. $\Gamma \vdash x$. Take $\phi \bullet \pi_x$, resp. $\bar{\pi}_x \bullet \phi$ to be ϕ. Let a be an atom such that $a \neq a\sigma$, resp. $a\bar{\sigma} \neq a$. Let ϕ be a derivation of $a, a \multimap a\sigma, \Gamma \vdash X$, resp. $\Gamma, a\bar{\sigma} \multimap a \vdash a$. Then ϕ permutes into a derivation

$$\frac{\overline{a \vdash a} \quad \overline{\overline{a\sigma, \Gamma \vdash X}}^{\gamma}}{a, a \multimap a\sigma, \Gamma \vdash X}\multimap_{\mathcal{L}} \text{, resp. } \frac{\overline{\overline{\Gamma \vdash a\bar{\sigma}}}^{\gamma} \quad \overline{a \vdash a}}{\Gamma, a\bar{\sigma} \multimap a \vdash a}\multimap_{\mathcal{L}} .$$

Take $\phi \bullet \pi_a$, resp. $\bar{\pi}_a \bullet \phi$ to be γ.
Induction. Let A and B be two propositions. Then, let ϕ be a derivation of $A \otimes B, \Pi_{A\otimes B}, \Gamma \vdash X$, resp. $\Gamma, \bar{\Pi}_{A\otimes B} \vdash A \otimes B$. We may assume that ϕ permutes into a derivation

$$\dfrac{\overline{\overline{A, \Pi_A, B, \Pi_B, \Gamma \vdash X}}^{\,\gamma}}{A \otimes B, \Pi_A, \Pi_B, \Gamma \vdash X}\otimes_{\mathcal{L}}\ , \text{ resp. } \dfrac{\overline{\overline{\Delta, \bar{\Pi}_A \vdash A}}^{\,\delta} \quad \overline{\overline{\Theta, \bar{\Pi}_B \vdash B}}^{\,\theta}}{\Delta, \bar{\Pi}_A, \Theta, \bar{\Pi}_B \vdash A \otimes B}\otimes_{\mathcal{R}}\ .$$

Take $\phi \bullet \pi_{A\otimes B}$, resp. $\bar{\pi}_{A\otimes B} \bullet \phi$ to be

$$\dfrac{\overline{\overline{A\sigma, B\sigma, \Gamma \vdash X}}^{\,(\gamma\bullet\pi_A)\bullet\pi_B}}{A\sigma \otimes B\sigma, \Gamma \vdash X}\otimes_{\mathcal{L}}\ , \text{ resp. } \dfrac{\overline{\overline{\Delta \vdash A\bar{\sigma}}}^{\,\bar{\pi}_A\bullet\delta} \quad \overline{\overline{\Theta \vdash B\bar{\sigma}}}^{\,\bar{\pi}_B\bullet\theta}}{\Delta, \Theta \vdash A\bar{\sigma} \otimes B\bar{\sigma}}\otimes_{\mathcal{R}}\ .$$

Let ϕ be a derivation of $A \multimap B, \Pi_{A\multimap B}, \Gamma \vdash X$, resp. $\Gamma, \bar{\Pi}_{A\multimap B} \vdash A \multimap B$. We may assume that ϕ permutes into a derivation

$$\dfrac{\overline{\overline{\Delta, \bar{\Pi}_A \vdash A}}^{\,\delta} \quad \overline{\overline{B, \Pi_B, \Theta \vdash X}}^{\,\theta}}{\Delta, \bar{\Pi}_A, A \multimap B, \Pi_B, \Theta \vdash X}\multimap_{\mathcal{L}}\ , \text{ resp. } \dfrac{\overline{\overline{A, \Pi_A, \Gamma, \bar{\Pi}_B \vdash B}}^{\,\gamma}}{\Pi_A, \Gamma, \bar{\Pi}_B \vdash A \multimap B}\multimap_{\mathcal{R}}\ .$$

Take $\phi \bullet \pi_{A\multimap B}$, resp. $\bar{\pi}_{A\multimap B} \bullet \phi$ to be

$$\dfrac{\overline{\overline{\Delta \vdash A\bar{\sigma}}}^{\,\bar{\pi}_A\bullet\delta} \quad \overline{\overline{B\sigma, \Theta \vdash X}}^{\,\theta\bullet\pi_B}}{\Delta, A\bar{\sigma} \multimap B\sigma, \Theta \vdash X}\multimap_{\mathcal{L}}\ , \text{ resp. } \dfrac{\overline{\overline{A\sigma, \Gamma \vdash B\bar{\sigma}}}^{\,(\bar{\pi}_B\bullet\gamma)\bullet\pi_A}}{\Gamma \vdash A\sigma \multimap B\bar{\sigma}}\multimap_{\mathcal{R}}\ .$$

■

P r o p o s i t i o n 14. *Let σ be a polarized substitution and A a proposition. Let ϕ be a derivation of $A\sigma, \Gamma \vdash X$, resp. of $\Gamma \vdash A\bar{\sigma}$. Then ϕ permutes into $(\phi \circ \pi_A) \bullet \pi_A$ resp. into $\bar{\pi}_A \bullet (\bar{\pi}_A \circ \phi)$.*

P r o o f. Obvious from the construction of $\phi \circ \pi_A$ and $(\phi \circ \pi_A) \bullet \pi_A$. ■

C o r o l l a r y 15. *Let ϕ and ϕ' be two derivations of $A\sigma, \Gamma \vdash X$, resp. of $\Gamma \vdash A\bar{\sigma}$. Then ϕ and ϕ' are inter-permutable iff $\phi \circ \pi_A$ and $\phi' \circ \pi_A$, resp. $\bar{\pi}_A \circ \phi$ and $\bar{\pi}_A \circ \phi'$ are inter-permutable.*

P r o o f. Assume that ϕ and ϕ' are inter-permutable. Then by proposition 12, so are $\phi \circ \pi_A$ and $\phi' \circ \pi_A$, resp. $\bar{\pi}_A \circ \phi$ and $\bar{\pi}_A \circ \phi'$. Assume that $\phi \circ \pi_A$ and $\phi' \circ \pi_A$, resp. $\bar{\pi}_A \circ \phi$ and $\bar{\pi}_A \circ \phi'$ are inter-permutable. Then by proposition 13, so are $(\phi \circ \pi_A) \bullet \pi_A$ and $(\phi' \circ \pi_A) \bullet \pi_A$, resp. $\bar{\pi}_A \bullet (\bar{\pi}_A \circ \phi)$ and $\bar{\pi}_A \bullet (\bar{\pi}_A \circ \phi')$. By proposition 14, so are ϕ and ϕ'. ■

4. Applications. 4.1. Deciding equivalence.

L e m m a 16. (Normal derivations ending by $\multimap_{\mathcal{L}}$) *Let ϕ be a normal derivation of $\Gamma, A \multimap B, \Delta \vdash C$. If $\Gamma \vdash A$ belongs to $N(\phi)$, then ϕ permutes into a normal derivation ϕ' ending by $\multimap_{\mathcal{L}}$ with left premise $\Gamma \vdash A$ and such that $N(\phi) = N(\phi')$.*

P r o o f. If $\Gamma \vdash A$ belongs to $N(\phi)$, then ϕ contains a subderivation ending by a $\multimap_{\mathcal{L}}$ inference with left premise a *minimal* derivation γ of $\Gamma \vdash A$. Let $\multimap_\gamma$ denote this inference. We proceed by induction on the height $\multimap_\gamma$ in ϕ.
Base. If ϕ is ending by $\multimap_\gamma$, then take ϕ' to be ϕ. Trivially, ϕ' is normal and $N(\phi) = N(\phi')$.
Induction. If ϕ is ending by a unary inference (r) with premise a normal derivation δ, then $N(\phi) = N(\delta)$ and $\Gamma \vdash A$ belongs to $N(\delta)$. By induction hypothesis δ permutes into a

normal derivation δ' ending by $\multimap_\gamma$ such that $N(\delta) = N(\delta')$. Whatever (r), as the table of permutations shows, $\multimap_\gamma$ may be pushed below (r). Take ϕ' to be the derivation obtained this way. Then ϕ' is normal and $N(\phi) = N(\phi')$. If ϕ is ending by a binary inference (r) with premises some normal derivations δ and θ, then $\Gamma \vdash A$ belongs to $N(\delta)$ or $N(\theta)$, since γ minimal. The previous argument applies in a similar way. Let's mention that if ϕ is ending by $\multimap_\mathcal{L}$, then $\Gamma \vdash A$ necessarily belongs to the normal set of the *right* premise. ■

P r o p o s i t i o n 17. (Inter-permutability of normal derivations) *Let ϕ and ϕ' be two normal derivations of the same sequent. Then ϕ and ϕ' are inter-permutable iff $N(\phi) = N(\phi')$.*

P r o o f. By structural induction on the height of the derivations.
Base. If ϕ is an axiom, then ϕ' is the same axiom. Trivially, ϕ and ϕ' are inter-permutable iff $N(\phi) = N(\phi')$.
Induction. If ϕ permutes into a derivation ending by $\otimes_\mathcal{L}$ or $\multimap_\mathcal{R}$ with premise a derivation γ, then ϕ' permutes into a derivation ending by $\otimes_\mathcal{L}$ or $\multimap_\mathcal{R}$ with premise a derivation γ' of the same sequent as γ (lemma 2). By induction hypothesis, γ and γ' are inter-permutable iff $N(\gamma) = N(\gamma')$. Then such is the case for ϕ and ϕ'. The same way, if ϕ permutes into a derivation ending by *wkn* or $\otimes_\mathcal{R}$ with premises some derivations γ and δ, then ϕ' permutes into a derivation ending by *wkn* or $\otimes_\mathcal{R}$ with premises some derivation γ' and δ' of the same sequents as γ and δ respectively (lemma 3). The previous argument applies. Finally, assume that ϕ and ϕ' are both ending by $\multimap_\mathcal{L}$ with the left premises γ and γ' respectively and right premises δ and δ' respectively. Let $\Gamma \vdash A$ and $\Gamma' \vdash A'$ be the end-sequents of γ and γ' respectively. Then $N(\phi) = N(\gamma) \cup \{\Gamma \vdash A\} \cup N(\delta)$ and $N(\phi') = N(\gamma') \cup \{\Gamma' \vdash A'\} \cup N(\delta')$. If $\Gamma \vdash A$ and $\Gamma' \vdash A'$ are the same sequent, then the same argument as previously applies. Else, assume that $N(\phi) = N(\phi')$. Then $\Gamma \vdash A$ belongs to $N(\phi')$ and by lemma 16, ϕ' permutes into a normal derivation ϕ'' ending by $\multimap_\mathcal{L}$ with left premise a minimal normal derivation γ'' of $\Gamma \vdash A$ and right premise a derivation δ'' of the same sequent as δ. Then $N(\phi'') = N(\phi') = N(\phi)$, so $N(\gamma'') = N(\gamma)$, $N(\delta'') = N(\delta)$, γ'' and γ are inter-permutable and so are δ'' and δ. As a consequence, ϕ'' and ϕ are inter-permutable. Finally, so are ϕ and ϕ'. Now, assume that ϕ and ϕ' are inter-permutable. The subderivations γ and γ' are minimal then δ and δ' permute into normal derivations ending by $\multimap_\mathcal{L}$ with left premises some derivations θ and θ' of $\Gamma' \vdash A'$ and $\Gamma \vdash A$ respectively and with right premises some derivations ζ and ζ' of the same sequent. Then $N(\phi) = N(\gamma) \cup \{\Gamma \vdash A\} \cup N(\theta) \cup \{\Gamma' \vdash A'\} \cup N(\zeta)$ and $N(\phi') = N(\gamma') \cup \{\Gamma' \vdash A'\} \cup N(\theta') \cup \{\Gamma \vdash A\} \cup N(\zeta)$. Since ϕ and ϕ' are inter-permutable, so are γ and θ', θ and γ', ζ and ζ'. Then $N(\gamma) = N(\theta')$, $N(\theta) = N(\theta')$ and $N(\zeta) = N(\zeta')$. Finally, $N(\phi) = N(\phi')$. ■

Thus, to decide the equivalence of two normal derivations, it is enough to compare their normal sets. However, for this result to be applicable, we should be able to associate a normal set to any non normal derivation and to construct this normal set without actually testing permutability. Due to the following proposition, it is possible.

P r o p o s i t i o n 18. *Every derivation ϕ permutes into a normal derivation $n(\phi)$ of the same sequent. Such a derivation is called a* normalized *derivation of ϕ. Proposition 18 allows us to define $N(\phi)$ as the normal set of any normalized derivation of ϕ. Constructing $N(\phi)$ is done essentially by analyzing its end-sequent S.*

Proof. By induction on the length of S.

Base. If $S = a \vdash a$ or $S = I \vdash I$, then ϕ is an axiom. Take $n(\phi)$ to be ϕ and $N(\phi)$ to be empty.

Induction. If $S = A \otimes B, \Gamma \vdash C$ (resp. $S = \Gamma \vdash A \multimap B$), then ϕ permutes into a derivation ϕ' ending by $\otimes_{\mathcal{L}}$ (resp. by $\multimap_{\mathcal{R}}$) with premise a derivation γ (lemma 2). Take $n(\phi)$ to be the derivation ending by $\otimes_{\mathcal{L}}$ (resp. by $\multimap_{\mathcal{R}}$) with premise $n(\gamma)$, and $N(\phi)$ to be $N(\gamma)$.

If $S = \Gamma, \Delta \vdash B$ (resp. $S = \Gamma, \Delta \vdash A \otimes B$) and if $T = \Gamma \vdash I$ (resp. $T = \Gamma \vdash A$) and $U = \Delta \vdash B$ are balanced, then ϕ permutes into a derivation ending by *wkn* (resp. by $\otimes_{\mathcal{R}}$) with left and right premises two derivations γ and δ (lemma 3). Take $n(\phi)$ to be the derivation ending by *wkn* (resp. by $\otimes_{\mathcal{R}}$) with left and right premises $n(\gamma)$ and $n(\delta)$ respectively and $N(\phi)$ to be $N(\gamma) \cup N(\delta)$.

Finally, assume that the previous cases are excluded. Then $S = \Gamma, A \multimap B, \Delta \vdash C$ and ϕ is ending by $\multimap_{\mathcal{L}}$ with left and right premises two derivations γ and δ of $T = \Gamma \vdash A$ and $U = B, \Delta \vdash C$ respectively. So $n(\gamma)$ is normal and do not permute into a derivation ending *wkn* or $\otimes_{\mathcal{L}}$ (so would be ϕ).

- If T can be presented as $\Gamma', A' \multimap B', \Delta' \vdash A$ with $\Gamma' \vdash A'$ in $N(\gamma)$, then $n(\gamma)$ permutes into a normal derivation ending by $\multimap_{\mathcal{L}}$ with premises some derivations γ' and δ' of $\Gamma' \vdash A'$ and $B', \Delta' \vdash A$ respectively (lemma 16). Let δ'' be the derivation ending by $\multimap_{\mathcal{L}}$ with left and right premises δ' and δ respectively. Then ϕ permutes into the derivation ending by $\multimap_{\mathcal{L}}$ with left and right premises γ' and δ'' respectively. Take $n(\phi)$ to be the derivation ending by $\multimap_{\mathcal{L}}$ with left and right premises γ' and $n(\delta'')$ respectively. Take $N(\phi)$ to be $N(\gamma') \cup \{\Gamma' \vdash A'\} \cup N(\delta'')$.
- If T can't be presented in such a way, then $n(\gamma)$ is minimal. Take $n(\phi)$ to be the derivation ending by $\multimap_{\mathcal{L}}$ with left and right premises $n(\gamma)$ and $n(\delta)$ respectively. Take $N(\phi)$ to be $N(\gamma) \cup \{\Gamma \vdash A\} \cup N(\delta)$.

∎ The construction of $N(\phi)$ is done in polynomial time. However, our work so far doesn't allow us to give the precise complexity of the algorithm.

4.2. Counting non equivalent derivations.. For every proposition A, let $A^{(n)}$ denote the nth "dual" of A, that is $A^{(0)} = A$ and $A^{(n+1)} = A^{(n)} \multimap I$. For every natural numbers p and q, let $\|A^{(2p)}, A^{(2q)}\|$ denote the number of non inter-permutable derivations of $A^{(2p)} \vdash A^{(2q)}$. In [16], Szabo claims that if A is not isomorphic to I, then $\|A^{(2p)}, A^{(2q)}\| = \binom{p+q-1}{p}\|A, A\|$. As noticed by Jay in [5], this statement is self-contradictory since it implies on the one hand that $\|A^{(4)}, A^{(4)}\| = \binom{3}{2}\|A, A\| = 3\|A, A\|$ and on the other hand that $\|A^{(4)}, A^{(4)}\| = \binom{1}{1}\|A^{(2)}, A^{(2)}\| = \binom{1}{1}\binom{1}{1}\|A, A\| = \|A, A\|$. Actually, Szabo's formula holds if $A^{(2)} \vdash A^{(0)}$ is not derivable. In this case, Jay's argument doesn't apply since $A^{(4)} \vdash A^{(2)}$ is derivable.

Proposition 19. *Let A be a proposition such that $A^{(2)} \vdash A^{(0)}$ is not derivable. For every natural numbers p and q, there are $\binom{p+q}{p}\|A, A\|$ non inter-permutable derivations of $A^{(2p)}, A^{(2q+1)} \vdash I$. As a consequence, $\|A^{(2p)}, A^{(2q)}\| = \binom{p+q-1}{p}\|A, A\|$.*

Remark 3. *The two non inter-permutable derivations of $A^{(4)}, A^{(3)} \vdash I$ where $\|A, A\| = 1$ are given in example 1.*

Proof. Let $D(p, q)$ denote the number on non inter-permutable derivations of $A^{(2p)}, A^{(2q+1)} \vdash I$. It is enough to show that $D(0, q) = D(p, 0) = \|A, A\|$ and $D(p + 1, q +$

$1) = D(p, q+1) + D(p+1, q)$. Indeed, assume these equalities. Then by double induction on p and q,

$$D(0, q) = \|A, A\| = \binom{0+q}{0}\|A, A\|, \tag{6}$$

$$D(p, 0) = \|A, A\| = \binom{p+0}{p}\|A, A\|, \tag{7}$$

$$\begin{aligned} D(p+1, q+1) &= D(p, q+1) + D(p+1, q) \\ &= \left(\binom{p+q+1}{p} + \binom{p+q+1}{p+1}\right)\|A, A\| \\ &= \binom{(p+1)+(q+1)}{p+1}\|A, A\|. \end{aligned} \tag{8}$$

Up to permutation, every derivation of $A^{(0)}, A^{(1)} \vdash I$ is ending by $\multimap_{\mathcal{L}}$. Due to balancedness, the right premise necessarily is $I \vdash I$. If A were isomorphic to a proposition $B \multimap I$, then $A^{(2)} \vdash A^{(0)}$ would be derivable. So the left premise is $A \vdash A$ and $D(0, 0) = \|A, A\|$. For the same reason, every derivation of $A^{(0)}, A^{(2(q+1)+1)} \vdash I$, resp. $A^{(2(p+1))}, A^{(1)} \vdash I$ is ending by $\multimap_{\mathcal{L}}$ with right premise $I \vdash I$ and left premise

$$\dfrac{\overline{A^{(0)}, A^{(2q+1)} \vdash I}}{A^{(0)} \vdash A^{(2(q+1))}}\multimap_{\mathcal{R}}, \text{ resp. } \dfrac{\overline{A^{(1)}, A^{(2p)} \vdash I}}{A^{(1)} \vdash A^{(2p+1)}}\multimap_{\mathcal{R}}.$$

Then $D(0, q+1) = D(0, q)$, resp. $D(p+1, 0) = D(p, 0)$ and $D(0, q) = D(p, 0) = \|A, A\|$. Every derivation of $A^{(2(p+1))}, A^{(2(q+1)+1)} \vdash I$ is ending by

$$\dfrac{\dfrac{\overline{A^{(2(q+1)+1)}, A^{(2p)} \vdash I}}{A^{(2(q+1)+1)} \vdash A^{(2p+1)}}\multimap_{\mathcal{R}} \quad \overline{I \vdash I}}{A^{(2(q+1)+1)}, A^{(2(p+1))} \vdash I}\multimap_{\mathcal{L}} \quad \text{or} \quad \dfrac{\dfrac{\overline{A^{(2(p+1))}, A^{(2q+1)} \vdash I}}{A^{(2(p+1))} \vdash A^{(2(q+1))}}\multimap_{\mathcal{R}} \quad \overline{I \vdash I}}{A^{(2(p+1))}, A^{((2q+1)+1)} \vdash I}\multimap_{\mathcal{L}}$$

Obviously, these derivations are not inter-permutable. Then $D(p+1, q+1) = D(p, q+1) + D(p+1, q)$. Finally, every derivation of $A^{(2p)} \vdash A^{(2q)}$ is ending by $\multimap_{\mathcal{R}}$ with premise $A^{(2p)}, A^{(2(q-1)+1)} \vdash I$. Then $\|A^{(2p)}, A^{(2q)}\| = \binom{p+q-1}{p}\|A, A\|$. ∎

Let's mention that our result generalizes the game semantics analysis by Murawski and Ong of the so-called "towers of units problem" (see [13]): if A does not contain any unit, then $\|A^{(n)}, A^{(n)}\| = \binom{n-1}{\lfloor n/2 \rfloor}\|A, A\| = \binom{n-1}{\lfloor n/2 \rfloor}$.

5. Conclusion and further work. We've given a precise justification in IMLL to the folk statement that two cut-free derivations are to be considered "the same" iff are inter-permutable. Then we used inter-permutability to handle some particular cases of coherence in free SMCC's.

However, the techniques developed for the study of categorical equivalence of derivations in IMLL have also direct applications to the verification of commutativity of diagrams in *non free* SMCC's (see [11]). Most of them can be applied to the case of equivalences containing $\equiv$ and corresponding to equality of morphisms in non-free categorical models. The results presented here are actually part of a larger working project of development of logical support for computations in commutative algebra.

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