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FIRST-ORDER LOGIC OF UNIFORM ATTACHMENT RANDOM GRAPHS WITH A GIVEN DEGREE[1]

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In this paper, we prove the first-order convergence law for the uniform attachment random graph with almost all vertices having the same degree. In the considered model, vertices and edges are introduced recursively: at time $m+1$, we start with a complete graph on $m+1$ vertices. At step $n+1$, vertex $n+1$ is introduced together with m edges joining the new vertex with m vertices chosen uniformly from those vertices of $1,\ldots,n$, whose degree is less than $d = 2m$. To prove the law, we describe the dynamics of the logical equivalence classes of the random graph using Markov chains. The convergence law follows from the existence of a limiting distribution of the considered Markov chain.

Keywords: Uniform attachment, convergence law, first-order logic, Markov chains.

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Introduction

In the present paper, we prove the first-order (FO for brevity) convergence law for uniform attachment random graphs with most vertices having a given degree using finite Markov chains.

FO sentences about graphs could include the following symbols: variables $x, y, x_1, \ldots$ (which represent vertices), logical connectives $\wedge, \vee, \neg, \Rightarrow, \Leftrightarrow$, two relational symbols (between variables) $\sim$ (adjacency) and $=$ (equality), brackets and quantifiers $\exists, \forall$ (see the formal definition in, e.g., [4]). The sequence G_n of random graphs obeys the FO convergence law if, for every FO sentence φ, $\Pr(\mathcal{G}_n \models \varphi)$ converges as $n \to \infty$. If the limit is either 0 or 1 for any formula, G_n obeys the zero-one law. If G_n obeys the zero-one law, it is trivial in terms of the FO logic in the sense that all properties are trivial on a typical large enough graph.

The FO logical laws usually proven using Ehrenfeucht-Fraïssé pebble game (see, e.g., [4, Chapter 11.2]). The connection between FO logic and the Ehrenfeucht-Fraïssé game is described in the following result.

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Theorem 1. *Duplicator wins the γ-pebble game on G and H in R rounds if and only if, for every FO sentence φ with at most γ variables and quantifier depth at most R, either φ is true on both G and H or it is false on both graphs.*

In particular, if for any $\epsilon > 0$ one could define a finite number of classes $\mathcal{A}_k$, $k = 1, ..., K$, of graphs such that Duplicator wins the pebble game on graphs of the same class and $\Pr(G_n \in \mathcal{A}_k) \to p_k$ for all $k = 1, ..., K$ with $\sum_{k=1}^{K} p_k > 1 - \epsilon$, then G_n obeys the FO convergence law.

Logic limit laws were studied on many different models, such as the binomial random graph ([9, 10]), random regular graphs ([2]), attachment models ([5, 6, 8]), etc. ([3, 11, 12]). In the present article, we are interested in recursive random graph models. One of the main arguments towards G_n not satisfying the zero-one law for such models is the existence of rare subgraphs, which appear with diminishing probability (and with high probability does not appear after some moment, see, e.g., [5]). In this case, even if G_n obeys the FO convergence law and not the zero-one law, it still could be asymptotically trivial in terms of some classes of sentences of the FO logic in a sense that the correctness of the property on a graph does not change after some moment (see, e.g., [6,8]). In such a case, the division on the classes $\mathcal{A}_k$, $k = 1, ..., K$, is based on subgraph of G_n on first N vertices for all $n > N$ (N depends on ϵ). The more difficult case is when the correctness of the property changes infinitely many times during the (graph) process. Such behavior is somewhat similar to the behavior of a Markov chain, which changes its state infinitely many times but still could have a limiting probability distribution. One of the main purposes of the present paper is to showcase such a connection between a graph obeying the FO convergence law and the existence of the limiting probability distribution for related Markov chains (in our case, such a chain would be finite).

Let us give a formal description of our model. Fix $m \in \mathbb{N}$, $m > 1$, and let $d = 2m$. We start with a complete graph G_{m+1} on $m + 1$ vertices. Then, on each step, we add a new vertex and draw m edges from it to different vertices chosen uniformly among vertices with a degree less than d. Note that the total degree (the sum of degrees of all vertices) of the graph G_n would be equal to $m(m-1) + 2m(n-m) = dn - m(m+1)$. In particular, the number of vertices of degree d in G_n is between $n - m(m+1)$ and $n-(m+1)$. Hence, it is always possible to draw m edges from a new vertex to different vertices (of a degree less than d). The model for $d > 2m$ was considered in [7], where a similar result was proven using Markov chains with infinitely many states, as well as the stochastic approximation technique.

The resulting graph is somewhat similar to a d-regular graph (i.e., for a graph with all vertices having degree d, the number of vertices or the degrees should be even for such a graph to exist). Note that regular graphs cannot be dynamic since drawing edges to vertices of a regular graph makes it non-regular. Hence, regular graphs are built separately for each n (where n is the number of vertices). Our model provides a way of building a graph with properties close to a regular graph through the dynamic procedure.

Let us formulate our main result.

Theorem 2. *G_n obeys the FO convergence law.*

1. Configuration of open vertices

For the graph G_n, consider its subgraph T_n, obtained in the following way. First, consider the subgraph of G_n on vertices that have degrees less than d, and, for each vertex, add to it the number of leaves equal to the difference between vertex degree in G_n and such a subgraph. Note that the set of all possible subgraphs obtained in such a way is finite. Also, T_{n+1} depends only on T_n and does not depend on n. As a result, T_n, $n \in \mathbb{N}$ form a Markov chain with a finite number of states (which corresponds to different types of subgraphs T_n).

Define $U^a(n)$ as a-neighborhood of vertices of T_n (i.e., the subgraph of G_n on vertices that are at distances at most a from T_n). There is a finite number of possible neighborhoods. Since transition between $U^a(n)$ and $U^a(n+1)$ depends only on the state of $U^a(n)$, $U^a(n)$ forms a finite Markov chain.

We need the following result about the existence of a limit distribution of a finite Markov chain (see, e.g., [1, Chapter 6] for more details on Markov chains and corresponding terminology).

Lemma 1. *Let A_n, $n \in \mathbb{N}$ be a finite Markov chain. If there exists $k \in \mathbb{N}$, and a state S_0 such that S_0 could be achieved from any state (including S itself) in precisely k steps, and any state could be achieved from S_0, then, for any state S, there is a constant $c_S > 0$, such that*

$$\Pr(A_n = S) \to c_S \quad \textit{as } n \to \infty. \tag{1}$$

Proof. Due to the existence of state S_0, the chain is irreducible (any state could be achieved from any state through S_0) and aperiodic (S_0 could be achieved from any state in both k and $k+1$ step since we could perform one random step). Since the Markov chain is finite, due to [1, Lemma 6.3.5], all states are non-null persistent. As a result, due to [1, Theorem 6.4.17], we get the statement of the lemma. □

We now prove the existence of a state that could be achieved from any state in the same number of steps for T_n and $U^a(n)$.

Lemma 2. *A state that consists of $m+1$ isolated vertices with m leaves is achievable from any state in precisely $m+1$ steps (we would call such a state a forest state).*

Proof. Since the total degree of G_n equals $2mn - m(m+1)$, it would take $m+1$ steps of drawing m edges to G_n from new vertices to make all vertices of G_n to have degrees equal to $d = 2m$ in G_{n+m+1}. Let us consider the following procedure. At each step, we would choose m vertices among vertices of G_n with the smallest degrees and draw edges to them. Let us prove by induction that this procedure gives us the desired result. Since the lowest degree in G_n is at least m, there are at least $m-1$ vertices in G_n that we could draw an edge into, so the first step is possible, and after it, the total degree of (vertices of) G_n (in G_{n+1}) equals $2mn - m^2$.

Let us prove by induction that each follow-up step is possible. To do so, we need to prove that in G_{n+1+k}, $k = 0, ..., m-1$, there are at least m vertices with degree less than d, the lowest degree is at most $d - m + k$, and the total degree equals $2mn - m(m-k)$. For $k = 0$, these conditions are satisfied. Assume they hold for k and prove that after the k-th step, they would be true for $k+1$. Since the total degree of G_n (in G_{n+1+k})

equals $2mn - m(m-k)$, and lowest degree is at least $d - (m-k)$, there are at most m vertices with degree $d-(m-k)$ and at least m vertices with degree less than d, so the step could be performed and the lowest degree of G_n in G_{n+k+2} would be at least $d - m + k + 1$. Also, the total degree of G_n would increase by m and be equal to $2mn - m(m-k) + m = 2mn - m(m-(k+1))$, and there would be at least m vertices of degree less than d.

As a result, we would get $m+1$ vertices of degree m that are connected only to vertices of $[n]$, which have degrees equal to d. □

For the chain $U^a(n)$, define the following state U^a_0. Consider $a+1$ consequent groups $A_1, ..., A_{m+1}$ of $m+1$ vertices, such that every vertex of A_i connected with and only with every vertices of A_{i-1} and A_{i+1}. Note that this state is achievable from any state in precisely $m(a+1)$ steps by repeating the procedure described in Lemma 2 $a+1$ times. Let $\mathcal{U}^a$ be the set of all possible neighborhoods $U^a(n)$, achievable from the state U^a_0. Therefore, due to Lemma 1, we get the following result.

Lemma 3. *For any graph $H \in \mathcal{U}^a$ there exists c_H such that*

$$\Pr\left(U^a(n) = H\right) \to c_H$$

almost surely, and

$$\sum_{H \in \mathcal{U}^a} c_H = 1.$$

□

2. Subgraphs on old vertices

In this section, we study a-neighborhoods of vertices that contain only vertices of degree d (in graphs G_n, for large enough n, once such a neighborhood is achieved on a vertex, it does not change afterward). We would call such neighborhoods crystallized. Note that only a finite number of such neighborhoods could be achieved. Moreover, for each possible configuration of $U^a_n(n)$, the probabilities of obtaining any achievable set of crystallized neighborhoods on vertices of T_n in bounded (by some constant C) number of steps depends only on graph $U^a_n(n)$ (and does not depend on n). Due to Lemma 3 the probability for $U^a_n(n)$ to have a given configuration separated from 0 for large enough n. Therefore, for a vertex n, the probability that its a-neighborhood at time $n+C$ would be crystallized and of a given achievable type is separated from 0. Hence, we get the following result.

Lemma 4. *For any k and any achievable from U^a_0 type of crystallized a-neighborhood with a high probability, there are at least k vertices with disjoint a-neighborhoods of that type in G_n.*

□

3. Convergence law

Fix $R \in \mathbb{N}$. Let $a = 3^R$. In this section, we provide division on classes $\mathcal{A}_k$ of graphs and prove the existence of the winning strategy in R rounds for a pair of graphs G_{n_1}, G_{n_2} within the same class (for large enough n_1, n_2). We would consider a division based on the type of the graph $U_n^a(n)$ and the type of graph on initial vertices. The configuration on $m+1$ initial vertices is not achievable again during the process. Therefore, to ensure that the configuration of $U_n^a(n)$ is achievable (from U_0^a), we need to make sure that the first $m+1$ vertices do not belong to $U_n^a(n)$. Since the number of vertices with a degree less than d does not exceed $m(m+1)$, the probability of increasing the degree of a given vertex is at least $\frac{1}{m+1}$. Hence, with high probability a-neighborhoods (we denote their union as $W^a(n)$) of the first $m+1$ vertices in G_n contains only vertices of degree d. Indeed, for $W^a(n)$ to be crystallized, we need to draw at most $(m+1)^{d+1}$ edges to the given vertices, so by large deviation estimates for Bernoulli random variables (see, e.g., [1, Theorem 5.11.4]), such probability is at least $1 - Ce^{-cn}$ for some constants $c, C > 0$. As a result, we get

Lemma 5. *For any n_0 with high probability all degrees of vertices from $[n_0]$ have degree d in G_n.*

Fix $\epsilon > 0$. Let N be such that with probability at least $1-\epsilon$, degrees of all vertices of $W^{2a}(N)$ in G_N equal to d (the same would be then true for all $n \geq N$). There is a finite number M of pairs of types of $W^{2a}(n)$ and $U_n^a(n)$. Let classes $\mathcal{A}_k$, $k = 1, \ldots, M$, be defined by pairs $(W^{2a}(n), U_n^a(n))$, such that degrees of all vertices of $W^{2a}(N)$ in G_N equal to d. Let us define the following properties of graphs G_{n_1}, G_{n_2}.

- Q1 G_{n_1} and G_{n_2} belong to the same class $\mathcal{A}_k$.
- Q2 For any achievable (from U_0^a) type of complete a-neighborhood of a vertex there are at least R vertices with non-intersecting a-neighborhoods of that type in G_n that does not intersect with $W^{2a}(n)$ and $U_n^a(n)$, $n = n_1, n_2$.

Note that the probability that for all $n > N$ graph G_n belongs to one of the classes $\mathcal{A}_k$ is at least $1 - \epsilon$.

Lemma 6. *If graphs G_{n_1}, G_{n_2} satisfy properties $Q1, Q2$, then Duplicator has a winning strategy on them.*

Proof. Let us consider the following strategy. For a vertex v and $r \in \mathbb{N}$, let $B_r(v)$ be its neighborhood of radius r. Let Spoiler be putting pebbles $x_1, ..., x_R$ and duplicator putting $y_1, ..., y_R$. We omit a reference to a graph in the notation for these balls – each time we use the notation, the host graph would be clear from the context. We need to make a strategy such that, on each step, subgraphs of G_{n_1} and G_{n_2} on pebbles are isomorphic. We will build the strategy by induction over i. Let us assume that $B_{2^{R-j+1}}(x_j)$ and $B_{2^{R-j+1}}(y_j)$ are the same (i.e., they isomorphic and keep correspondence between pebbles) for $j < i$.

1. If $d(x_i, [m+1]) < 2^{R-i+1}$ then we put $y_i = x_i$. Note that their neighborhoods belong to $W^a(n)$ and, hence, are the same.

2. If x_i belongs to one of $U_n^{2^{R-i+1}}(n)$, $n = n_1, n_2$, (without loss of generality assume it is $U_{n_1}^{2^{R-i+1}}(n_1)$), it's neighborhood belongs to $U_{n_1}^a(n_1)$. Since $U_{n_1}^a(n_1)$ and $U_{n_2}^a(n_2)$

are the same, we could choose y_i in $U^a_{n_2}(n_2)$ that corresponds to x_i such that their neighborhoods would be the same.

3. If x_i does not belong to either one of $W^{2^{R-i+1}}(n)$, $U_n^{2^{R-i+1}}(n)$, $n = n_1, n_2$, our goal is to choose y_i in a way that its 2^{R-i+1} neighborhood would be exactly the same as of x_i. If there are pebbles in $B_{2^{R-i+1}}(x_i)$ (let j be the lowest index of such a pebble), then $B_{2^{R-i+1}}(x_i)$ belongs to $B_{2^{R-j+1}}(x_j)$ (or $B_{2^{R-j+1}}(y_j)$), and, therefore, x_i corresponds to the vertex in $B_{2^{R-j+1}}(y_j)$ (or $B_{2^{R-j+1}}(x_j)$), which we put as y_i. Now consider the case when there are no pebbles in $B_{2^{R-i+1}}(x_i)$. If it does not belong to the neighborhood of one of the previous pebbles, there are at least $R-i+1$ vertices with the same a-neighborhood (that does not interact with previously chosen vertices), and we choose any of them as y_i. If it belongs to the neighborhood of one of the previous pebbles, then it belongs along with its neighborhood to a wider a-neighborhood of one of the previous pebbles and, hence, corresponds to a vertex y_i in the same a-neighborhood in the other graph. □

As a result, if property Q2 holds for all large enough pairs n_1, n_2, then (due to Theorem 1) for each FO sentence, it is either true or false on G_n (for large enough n) depending only on the class $\mathcal{A}_i$, to which G_n belongs. Hence, Theorem 2 follows from

Lemma 7. *For any $R \in \mathbb{N}$ and any $\epsilon > 0$ there is $N \in \mathbb{N}$, and numbers $p_i > 0$, $i \in [M]$, $\sum_{i=1}^M p_i \geq 1-\epsilon$, such that*

- *With probability at least $1-\epsilon$ for all $n_1 > n_2 > N$ the pair (G_{n_1}, G_{n_2}) has the property* Q2*;*
- *for every $i \in [M]$, $\lim_{n\to\infty} \Pr(G_n \in \mathcal{A}_i) = p_i$.*

Proof. The first part follows from Lemma 4 and the fact that the number of crystallized neighborhoods of the given type in G_n is non-decreasing over n.

The second part follows from Lemma 5 and Lemma 3. □

Conclusion

In the present paper we demonstrated how Markov chains could be used to prove convergence laws about first-order logic on graphs. The model considered in the article was relatively simple, so we were able to highlight main steps of the prove without large amount of technical details. More general models would require infinite Markov chains with more complex transition, which would be much harder to study, but general structure of the prove should remain the same.

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ЛОГИКА ПЕРВОГО ПОРЯДКА НА ГРАФАХ РАВНОМЕРНОГО ПРИСОЕДИНЕНИЯ С ЗАДАННОЙ СТЕПЕНЬЮ ВЕРШИН

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В статье доказывается закон сходимости для логики первого порядка на случайных графах с равномерным присоединением вершин, в которых почти все вершины имеют одинаковую степень. В рассматриваемой модели вершины и ребра вводятся рекурсивно: в момент времени $m+1$ мы начинаем с полного графа на $m+1$ вершине. На шаге $n+1$ добавляется вершина $n+1$ вместе с m ребрами, соединяющими новую вершину с m вершинами, выбранными равновероятно из тех вершин из $1, \dots, n$, степень которых меньше $d = 2m$. Для доказательства закона мы описываем динамику классов логической эквивалентности случайного графа с помощью цепей Маркова. Закон сходимости следует из существования предельного распределения рассматриваемой цепи Маркова.

Ключевые слова: равномерное присоединение, логика первого порядка, законы сходимости, Марковские цепи.

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