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INTEGRO CUBIC SPLINES AND THEIR APPROXIMATION PROPERTIES[1]

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Показано, что трудности при построении интегрокубического сплайна, предложенного в работе Х. Бехфоруз [1], могут быть преодолены, используя B-представление. Также рассмотрены свойства приближения такого сплайна.

It is shown that the difficulties in constructing the integro cubic spline proposed by H. Behforooz [1] may be overcome using its B-representation. The approximation properties of such a spline are also considered.

Ключевые слова: интегрокубические сплайны, B-представление, свойства приближения.
Keywords: integro cubic splines, B-representation, approximation properties.

1. Introduction

In [1] H. Behforooz introduced integro cubic splines, and the accuracy of this type splines was shown by numerical experiments. Motivation of construction of such splines was also explained in [1] by numerous practical applications. To construct the integro cubic splines proposed in [1], besides two end conditions, also one additional/or third end condition is needed that seems to be unnaturally. He pointed out that to construct the integro cubic splines in terms of the second derivative with any end conditions, one had to solve a system of linear equations with a full matrix of higher order. In this paper we show that using B-representation of cubic splines one can overcome the difficulties arising in H. Behforooz approaches. We also prove that the unique integro cubic spline exists under the appropriate given end condition and the algorithm of constructing such a spline leads to solving a tridiagonal system. Approximation properties of the splines constructed using B-representation are also considered.

2. Preliminaries

Suppose that the interval [a,b] is partitioned by the following $k+1$ equally spaced points:

$$a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b, \qquad (1)$$

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such that $x_i = a + ih$, for $i = 0, 1, \dots, k$ with $h = (b - a)/k$. Assume that the function values $y_i = y(x_i)$ are not given but integrals of $y = y(x)$ are known on k intervals $[x_{i-1}, x_i]$ and they are equal to

$$\int_{x_{i-1}}^{x_i} y(x)dx = I_i, \ i = 1(1)k. \tag{2}$$

The cubic splines $S(x) \in C^2[a, b]$ are called integro cubic ones [1], if

$$\int_{x_{i-1}}^{x_i} S(x)dx = \int_{x_{i-1}}^{x_i} y(x)dx = I_i, \quad i = 1(1)k. \tag{3}$$

For simplicity, we will use the notations: $y_i = y(x_i)$, $S_i = S(x_i)$, $m_i = S^{'}(x_i)$ and $M_i = S^{''}(x_i)$. If we use a first derivative representation of $S(x) \in C^2[a, b]$, then it is easy to show that the conditions (3) lead to

$$h^2(m_{i-1} - m_i) + 6h(S_{i-1} + S_i) = 12I_i, \ i = 1(1)k, \tag{4}$$

$$h^2(m_i - m_{i+1}) + 6h(S_i + S_{i+1}) = 12I_{i+1}, \ i = 0(1)k - 1. \tag{5}$$

From (4), (5) and from the well-known consistency relations

$$m_{i-1} + 4m_i + m_{i+1} = \frac{3}{h}(S_{i+1} - S_{i-1}), \quad i = 1(1)k - 1, \tag{6}$$

it follows that [1]

$$m_{i-1} + 10m_i + m_{i+1} = \frac{12}{h^2}(I_{i+1} - I_i), \ i = 1(1)k - 1. \tag{7}$$

In order to construct the cubic spline S using Eq. (7) and solve them for $k+1$ unknowns $m_0, m_1, \dots, m_k$, we need (as usual) two additional equations. Suppose that $y'(a) = \alpha$ and $y'(b) = \beta$ are given. Then by setting $m_0 = \alpha$ and $m_k = \beta$, we can solve easily the following $(k - 1)$ by $(k - 1)$ linear tridiagonal equations to obtain a unique set of solutions for $m_1, m_2, \dots, m_{k-1}$:

$$\begin{cases} 10m_1 + m_2 = b_1 - \alpha, \\ m_{i-1} + 10m_i + m_{i+1} = b_i, \ i = 2(1)k - 2, \\ m_{k-1} + 10m_k = b_{k-1} - \beta, \end{cases} \tag{8}$$

where $b_i = \frac{12}{h^2}(I_{i+1} - I_i)$. After finding $m_0, m_1, \dots, m_k$ from (8), we can use (4) or (5) to compute $S_0, S_1, \dots, S_k$. But we need another additional given value for $y(a)$ or $y(b)$. If the additional (third) end condition $y_0 = y(a)$ is not given and $y(a)$ is not available, in [1] it was proposed to use the relations

$$S_1 - S_0 = hm_0 \quad \text{or} \quad S_1 - S_0 = hm_1 \tag{9}$$

as an additional equation. However, in this case we lose the order of accuracy of spline, due to (9).

If we use a second derivative representation of $S(x)$, then it is easy to show that the conditions (3) lead to

$$-\frac{h^3}{24}(M_{i-1}+M_i)+\frac{h}{2}(S_{i-1}+S_i)=I_i,\ i=1(1)k, \tag{10}$$

$$-\frac{h^3}{24}(M_i+M_{i+1})+\frac{h}{2}(S_i+S_{i+1})=I_{i+1},\ i=0(1)k-1. \tag{11}$$

Unlike the first derivative representation, here, we cannot eliminate S's between (10), (11) and the consistency relations

$$M_{i-1}+4M_i+M_{i+1}=\frac{6}{h^2}(S_{i-1}-2S_i+S_{i+1}),\ i=1(1)k-1 \tag{12}$$

to obtain a relation similar to (7) without S's. So, to construct $S(x)$ using the second derivative representation with any end conditions, we have to solve a system of linear equations with a full matrix of order $(2k+2)$ by $(2k+2)$. It should be pointed out that the above mentioned conclusions are the main results of paper [1].

3. Integro cubic spline with B-representation

Now we proceed to use the B-representation of cubic spline $S(x)$ of class $C^2[a,b]$. To do this, the partition of $[a,b]$ is extended to the left and right sides by equally spaced knots

$$x_{-3}<x_{-2}<x_{-2}<x_0,\ x_k<x_{k+1}<x_{k+2}<x_{k+3}.$$

Then we have [2,3]

$$S(x)=\sum_{j=-1}^{k+1}\alpha_j B_j(x), \tag{13}$$

where $B_j(x)$ is a normalized cubic B-splines with compact support $[x_{j-2},x_{j+2}]$. The coefficients of expansion (13) are given by [3]:

$$\alpha_j=S_j+\frac{h_j-h_{j-1}}{3}m_j-\frac{h_jh_{j-1}}{6}M_j\quad j=0,1,\ldots,k. \tag{14}$$

In case of a uniform partition the formula (14) becomes

$$\alpha_j=S_j-\frac{h^2}{6}M_j,\ j=0,1,\ldots,k. \tag{15}$$

Also from (13) it follows:

$$S_i=\frac{\alpha_{i+1}+4\alpha_i+\alpha_{i-1}}{6}, \tag{16}$$

$$m_i=\frac{\alpha_{i+1}-\alpha_{i-1}}{2h}, \tag{17}$$

$$M_i=\frac{\alpha_{i+1}-2\alpha_i+\alpha_{i-1}}{h^2}, \tag{18}$$

where $i=0,1,\ldots,k$.

We will show that the difficulties mentioned above may be overcome by using a B-representation (13), instead of a second derivative representation. In order to show that, we rewrite the relations (10) and (11) in term of expansion coefficients α_i

$$\alpha_{i-1} + \alpha_i + \frac{h^2}{12}(M_{i-1} + M_i) = \frac{2}{h} I_i, \quad i = 1(1)k, \tag{19}$$

$$\alpha_i + \alpha_{i+1} + \frac{h^2}{12}(M_i + M_{i+1}) = \frac{2}{h} I_{i+1}, \quad i = 0(1)k-1. \tag{20}$$

By adding (19) and (20), we get

$$\alpha_{i-1} + 2\alpha_i + \alpha_{i+1} + \frac{h^2}{12}(M_{i-1} + 2M_i + M_{i+1}) = \frac{2}{h}(I_i + I_{i+1}), \quad i = 1(1)k-1 \tag{21}$$

or

$$S_{i-1} + 2S_i + S_{i+1} - \frac{h^2}{12}(M_{i-1} + 2M_i + M_{i+1}) = \frac{2}{h}(I_i + I_{i+1}), \quad i = 1(1)k-1. \tag{22}$$

If we use a notation

$$d_i = \alpha_i + S_i, \quad i = 0(1)k, \tag{23}$$

then it is easy to check that the relations (19) and (20) are equivalent to

$$d_{i-1} + d_i = \frac{4}{h} I_i, \quad i = 1(1)k, \tag{24}$$

and

$$d_i + d_{i+1} = \frac{4}{h} I_{i+1}, \quad i = 0(1)k-1, \tag{25}$$

respectively. Adding (24) and (25), we get

$$d_{i-1} + 2d_i + d_{i+1} = \frac{4}{h}(I_i + I_{i+1}), \ i = 1(1)k-1. \tag{26}$$

From (26) it is clear that $d_1, d_2, \ldots, d_{k-1}$ are determined by solving this tridiagonal system, if d_0 and d_k are given. However, instead of solving the tridiagonal system we can find all d_i using the following formulas

$$d_i = (-1)^i d_0 + \frac{4}{h} \sum_{j=1}^{i} (-1)^{i+j} I_j, \quad i = 1(1)k \tag{27}$$

or

$$d_i = (-1)^i d_k + \frac{4}{h} \sum_{j=i+1}^{k} (-1)^{i+j+1} I_j, \ i = k-1, k-2, \ldots, 0, \tag{28}$$

that immediately followed from (24) and (25). We now want to show how d_0 and d_k can be find. What was used to find them? To do this we consider a useful identity

$$\alpha_{i-1} + 2\alpha_i + \alpha_{i+1} = 4\alpha_i + h^2 M_i. \tag{29}$$

Using (29) we can rewrite (21) as

$$\alpha_i + \frac{h^2}{48}(M_{i-1} + 14M_i + M_{i+1}) = \frac{I_i + I_{i+1}}{2h}. \tag{30}$$

Since $M_{i+1} + M_{i-1} = 2M_i + h(S'''_{i+0} - S'''_{i-0})$, then from (30) it immediately follows that

$$\alpha_{i-1} + \alpha_i + \alpha_{i+1} = \frac{3}{2h}(I_i + I_{i+1}) - \frac{h^3}{16}(S'''_{i+0} - S'''_{i-0}),\ i = 1(1)k-1. \tag{31}$$

The last term in the right-hand side of (31) is small and it can be neglected. As a result we obtain an approximate formula

$$\alpha_{i-1} + \alpha_i + \alpha_{i+1} = \frac{3}{2h}(I_i + I_{i+1}),\ i = 1(1)k-1. \tag{32}$$

Taking into account (32) for $i = 1$, we obtain

$$S_1 + \frac{h^2}{6}M_1 = \frac{\alpha_0 + \alpha_1 + \alpha_2}{3} = \frac{I_1 + I_2}{2h}.$$

Therefore we have

$$d_1 = \alpha_1 + S_1 = 2S_1 - \frac{h^2}{6}M_1 = \frac{I_1 + I_2}{h} - \frac{h^2}{2}M_1,$$

in which we have used (15). From the last formula and (24) we get

$$d_0 = \frac{4}{h}I_1 - d_1 = \frac{3I_1 - I_2}{h} + \frac{h^2}{2}M_1. \tag{33}$$

Analogously we have

$$d_k = \frac{3I_k - I_{k-1}}{h} + \frac{h^2}{2}M_{k-1}. \tag{34}$$

Thus the quantities d_0 and d_k are determined by formula (33) and (34), respectively, if M_1 or M_{k-1} are known. If $M_1 = y''(x_1)$ and $M_{k-1} = y''(x_{k-1})$ are not given or $y''(x_1)$ and $y''(x_{k-1})$ are not available, we can use simple formulas

$$d_0 = \frac{3I_1 - I_2}{h};\quad d_k = \frac{3I_k - I_{k-1}}{h}. \tag{35}$$

Of course, in this case the approximation order reduces by two.

From (23) it follows that

$$\alpha_{i-1} + 10\alpha_i + \alpha_{i+1} = 6d_i, \quad i = 0(1)k. \tag{36}$$

On the other hand, the relations (24) in term of α_i are rewritten as

$$\alpha_{i-2} + 11\alpha_{i-1} + 11\alpha_i + \alpha_{i+1} = \frac{24}{h}I_i, \quad i = 1(1)k. \tag{37}$$

From (37) and (32) with $i = 1$ it follows

$$\alpha_{-1} + 10\alpha_0 + 10\alpha_1 = \frac{3}{2h}(15I_1 - I_2).$$

From the last equality and from (36) we obtain

$$\alpha_1 = \frac{15I_1 - I_2}{6h} - \frac{2}{3}d_0 = \frac{I_1 + I_2}{2h} - \frac{h^2}{3}M_1. \tag{38}$$

Analogously we find

$$\alpha_{k-1} = \frac{15I_k - I_{k-1}}{6h} - \frac{2}{3}d_k. \tag{39}$$

When $d_0, d_1, \ldots, d_k$ and α_1, α_{k-1} are known the coefficients $\alpha_2, \alpha_3, \ldots, \alpha_{k-2}$ are determined from the system

$$\begin{cases} 10\alpha_2 + \alpha_3 = 6d_2 - \alpha_1 \\ \alpha_{i-1} + 10\alpha_i + \alpha_{i+1} = 6d_i, \; i = 3(1)k-3, \\ \alpha_{k-3} + 10\alpha_{k-2} = 6d_{k-2} - \alpha_{k-1}, \end{cases} \tag{40}$$

which follows from (36). After solving the system of linear equations (40) the remainder coefficients α_{-1}, α_0 and α_k, α_{k+1} will be determined from (36) for $i = 0, 1$ and $i = k-1, k$ respectively. Thus, we find all the coefficients α_i of B-representation of integro cubic spline. The values of this spline and its first two derivatives at the knots x_i are determined by formula (16), (17) and (18). The values of integro cubic spline at any point $\bar{x} \in [a, b]$ different from knots x_i are given by

$$S(\bar{x}) = \sum_{j=-1}^{k+1} \alpha_j B_j(\bar{x}),$$

in which the explicit formula for B_i-splines have been used. Thus the construction of the integro cubic spline to approximate the function $y(x)$ leads to solving the $(k-3)$ by $(k-3)$ tridiagonal linear system (40). As mentioned above, when we use the second derivative representation, the construction of the integro cubic spline require to solve the system of linear equations with a full matrix of order $(2k+2)$ by $(2k+2)$. The main advantage of our approach is to use of B-representation.

When the first derivative end conditions are given the algorithm of construction of spline consists of two steps: first, as before, the system of equations (8) is solved. Once $m_0, m_1, \ldots, m_k$ are known, we can use formula (17) to compute the expansion coefficients ,i.e,

$$\alpha_{i+1} - \alpha_{i-1} = 2hm_i, \; i = 0(1)k. \tag{41}$$

It is easy to show that the linear combination of Eq.(37) with $i = 1$ and (41) with $i = 0$ and $i = 1$ yields

$$\alpha_0 + \alpha_1 = \frac{2}{h}I_1 + \frac{h}{6}(m_0 - m_1). \tag{42}$$

On the other hand, from approximate formula (32) with $i = 1$ and from (41) with $i = 1$ it follows that

$$2\alpha_0 + \alpha_1 = \frac{3}{2h}(I_1 + I_2) - 2hm_1. \tag{43}$$

As a consequence of (42) and (43), we have

$$\alpha_0 = \frac{3I_2 - I_1}{2h} - \frac{h}{6}(11m_1 + m_0), \tag{44}$$

$$\alpha_1 = \frac{5I_1 - 3I_2}{2h} + \frac{h}{3}(m_0 + 5m_1). \tag{45}$$

All other coefficients α_i are determined using (41) for $i = 1(1)k$ and $\alpha_{-1} = \alpha_1 - 2hm_0$. Thus, when using B-representation of cubic splines we did not need another third end conditions, unlike the using first derivative representation in [1].

4. Approximation properties of integro cubic splines

Now we investigate the approximation properties of integro cubic splines. First of all, we will derive some useful formulas from (3). We assume $y(x)$ is a six-times continuously differentiable function on interval [a,b]. Then using the Taylor expansion of function $y(x)$ at a x_{i-1} we get

$$y(x) = y_{i-1} + y'_{i-1}(x - x_{i-1}) + \frac{y''_{i-1}}{2}(x - x_{i-1})^2 +$$

$$\frac{y'''_{i-1}}{3!}(x - x_{i-1})^3 + \frac{y^{(4)}_{i-1}}{4!}(x - x_{i-1})^4 + O(h^5), \quad x \in [x_{i-1}, x_i]$$

in (3), we get

$$\frac{I_i}{h} = \sum_{k=0}^{4} \frac{y^{(k)}_{i-1}}{(k+1)!}h^k + O(h^5). \tag{46}$$

Analogously using the expansion of $S(x)$ at a x_{i-1}

$$S(x) = S_{i-1} + m_{i-1}(x - x_{i-1}) + \frac{M_{i-1}}{2}(x - x_{i-1})^2 + \frac{S'''_{i-1+0}}{3!}(x - x_{i-1})^3$$

in (3), we get

$$\frac{I_i}{h} = S_{i-1} + \frac{h}{2}m_{i-1} + \frac{h^2}{3!}M_{i-1} + \frac{h^3}{4!}S'''_{i-1+0}, \tag{47}$$

where $S'''_{i-1+0} = S'''(x_{i-1} + 0)$. From (46) and (47) it follows that

$$S_{i-1} - y_{i-1} + \frac{h}{2}(m_{i-1} - y'_{i-1}) + \frac{h^2}{3!}(M_{i-1} - y''_{i-1}) + \frac{h^3}{4!}(S'''_{i-1+0} - y'''_i) = O(h^4), \ i = 1(1)k.$$

Replacing $i - 1$ by i in the last relation one can rewrite it as:

$$S_i - y_i + \frac{h}{2}(m_i - y'_i) + \frac{h^2}{3!}(M_i - y''_i) + \frac{h^3}{4!}(S'''_{i+0} - y'''_i) = O(h^4) \quad i = 0(1)k-1. \tag{48}$$

Analogously, using Taylor expansion of function $y(x)$ and $S(x)$ at $x = x_i$, we get

$$S_i - y_i - \frac{h}{2}(m_i - y'_i) + \frac{h^2}{3!}(M_i - y''_i) - \frac{h^3}{4!}(S'''_{i-0} - y'''_i) = O(h^4), \ i = 1(1)k. \tag{49}$$

By adding and substracting (48) and (49) we get

$$S_i - y_i + \frac{h^2}{6}(M_i - y''_i) + \frac{h^4}{48}(\frac{S'''_{i+0} - S'''_{i-0}}{h}) = O(h^4), \ i = 1(1)k-1 \tag{50}$$

and

$$m_i - y_i^{'} + \frac{h^2}{12}(\frac{S_{i+0}^{'''} + S_{i-0}^{'''}}{2} - y_i^{'''}) = O(h^3),\ i = 1(1)k-1. \tag{51}$$

On the other hand, using Taylor expansion of $y(x) \in C^6[a,b]$ in (3), one can get

$$I_i/h = y_i - \frac{h}{2}y_i^{'} + \frac{h^2}{6}y_i^{''} - \frac{h^3}{24}y_i^{'''} + \frac{h^4}{5!}y_i^{IV} - \frac{h^5}{6!}y_i^{V} + O(h^6), \tag{52}$$

$$I_{i+1}/h = y_i + \frac{h}{2}y_i^{'} + \frac{h^2}{6}y_i^{''} + \frac{h^3}{24}y_i^{'''} + \frac{h^4}{5!}y_i^{IV} + \frac{h^5}{6!}y_!^{V} + O(h^6). \tag{53}$$

Adding and substracting (52) and (53) we get

$$\frac{I_i + I_{i+1}}{2h} = y_i + \frac{h^2}{3!}y_i^{''} + \frac{h^4}{5!}y_i^{IV} + O(h^6),\ i = 1(1)k \tag{54}$$

and

$$\frac{I_{i+1} - I_i}{h^2} = y_i^{'} + \frac{h^2}{12}y_i^{'''} + \frac{h^4}{360}y_i^{V} + O(h^5),\ i = 1(1)k. \tag{55}$$

In order to derive estimations for $S_i^{(r)} - y_i^{(r)}$, for $r = 0, 1, 2, 3$ we will use the equations (26) and end conditions

$$M_1 = y_1^{''} \quad \text{and} \quad M_{k-1} = y_{k-1}^{''}. \tag{56}$$

Since d_0 and d_k are given by formulas (33) and (34) respectively, the equations (26) can be rewritten as

$$\begin{cases} 2d_1 + d_2 = \frac{4}{h}(I_1 + I_2) - d_0, \\ d_{i-1} + 2d_i + d_{i+1} = \frac{4}{h}(I_i + I_{i+1}),\ i = 2(1)k-2, \\ d_{k-2} + 2d_{k-2} = \frac{4}{h}(I_{k-1} + I_k) - d_k. \end{cases} \tag{57}$$

If we use a notation

$$\theta_i = 2(S_i - y_i) - \frac{h^2}{6}(M_i - y_i^{''}),\ i = 1(1)k-1 \tag{58}$$

then from (57) it immediately follows that

$$\begin{cases} 2\theta_1 + \theta_2 = c_1, \\ \theta_{i-1} + 2\theta_i + \theta_{i+1} = c_i,\ i = 2(1)k-2, \\ \theta_{k-2} + 2\theta_{k-1} = c_{k-1}, \end{cases} \tag{59}$$

where

$$c_1 = \frac{4}{h}(I_1 + I_2) - d_0 - 2(2y_1 + y_2) + \frac{h^2}{6}(2y_1^{''} + y_2^{''}),$$

$$c_i = \frac{4}{h}(I_i + I_{i+1}) - 2(y_{i-1} + 2y_i + y_{i+1}) + \frac{h^2}{6}(y_{i-1}^{''} + 2y_i^{''} + y_{i+1}^{''}),\quad i = 2(1)k-2,$$

$$c_{k-1} = \frac{4}{h}(I_{k-1} + I_k) - d_k - 2(2y_{k-1} + y_{k-2}) + \frac{h^2}{6}(2y_{k-1}^{''} + y_{k-2}^{''}). \tag{60}$$

Lemma 4.1 Assume that d_0 and d_k are given by (33) and (34), in which M_1 and M_{k-1} defined by (56). If $y(x) \in C^4[a,b]$, then

$$\theta_i = O(h^4),\ i = 1(1)k-1. \tag{61}$$

Proof: Using the Taylor expansion of function $y(x) \in C^4(a,b)$ at a point x_i and (54) we obtain easily

$$c_i = O(h^4),\ i = 2(1)k-2. \tag{62}$$

By analogy, using (54), (33), (34) and the Taylor expansion of function $y(x)$ at the points x_1 and x_{k-1}, one can easily obtain

$$c_1 = O(h^4),\ c_{k-1} = O(h^4). \tag{63}$$

Since the matrix of the system of linear equations (59) is diagonally dominant, it has a unique solution $(\theta_1, \theta_2, \ldots, \theta_{k-1})$. According to (62), (63), the estimation (61) is fulfilled. □

Lemma 4.2 For d_i we have an estimation

$$d_i = 2y_i - \frac{h^2}{6}y_i^{''} + O(h^4), \quad i = o(1)k. \tag{64}$$

Proof. From Lemma 4.1 and (23), (58) follows the estimation (64) for $i = 1(1)k-1$. By virtue of (24) we have

$$d_0 = \frac{4}{h}I_1 - d_1.$$

Using (64) and (52) for $i = 1$ in the last equality, we get

$$d_0 = 2y_0 - \frac{h^2}{6}y_0^{''} + O(h^4).$$

,i.e., the estimate (64) is proved for $i = 0$. Analogously using (24) for $i = k$ and (64), (52) for $i = k$ we obtain (64) for $i = k$. □

Remark 4.3 More detailed analysis show that

$$d_i = 2y_i - \frac{h^2}{6}y_i^{''} + \frac{h^4}{60}y_i^{IV} + O(h^6),\ i = 1(1)k-1 \tag{65}$$

provided $y(x) \in C^6[a,b]$. We now ready to prove the main result.

Theorem 4.4. Let $S(x)$ be an integro cubic spline to approximate the function $y(x) \in C^4[a,b]$, satisfying a given conditions (3) and end conditions (56). Then for the coefficients of the B-representation of this spline $S(x)$ we have

$$\alpha_{-1} = y_0 - hy_0^{'} + \frac{h^2}{3}y_0^{''} + O(h^4), \tag{66}$$

$$\alpha_i = y_i - \frac{h^2}{6}y_i^{''} + O(h^4),\ i = 0(1)k, \tag{67}$$

$$\alpha_{k+1} = y_k + hy_k^{'} + \frac{h^2}{3}y_k^{''} + O(h^4). \tag{68}$$

Proof. In section 3 we show that the coefficients of integro cubic spline S are defined by the system (36), i.e.,

$$10\alpha_2+\alpha_3=6d_2-\alpha_1,$$

$$\alpha_{i-1}+10\alpha_i+\alpha_{i+1}=6d_i,\ i=3(1)k-3,$$

$$\alpha_{k-3}+10\alpha_{k-2}=6d_{k-2}-\alpha_{k-1}.$$

The right-hand side of the last system is defined by (27) and (38), (39). If we use a notation

$$\omega_i=\alpha_i-y_i+\frac{h^2}{6}y_i^{''},\quad i=2(1)k-2 \tag{69}$$

then from the last system it follows that

$$\begin{cases} 10\omega_2+\omega_3=z_2, \\ \omega_{i-1}+10\omega_i+\omega_{i+1}=z_i,\ i=3(1)k-3, \\ \omega_{k-3}+10\omega_{k-2}=z_{k-2}, \end{cases} \tag{70}$$

where

$$z_2=6d_2-\alpha_1-10y_2-y_3+\frac{h^2}{6}(y_2^{''}+y_3^{''}),$$

$$z_i=6d_i-(y_{i-1}+10y_i+y_{i+1})+\frac{h^2}{6}(y_{i-1}^{''}+10_i^{''}+y_{i+1}^{''}),\ i=3(1)k-3, \tag{71}$$

$$z_{k-2}=6d_{k-2}-\alpha_{k-1}-10y_{k-2}-y_{k-3}+\frac{h^2}{6}(y_{k-2}^{''}+y_{k-3}^{''}).$$

Using Taylor expansion of function $y(x)\in C^4[a,b]$ at a point x_i and (64) we get

$$z_i=O(h^4),\quad i=3(1)k-3.$$

Analogously using (64), (38) and (39) we have $z_2=O(h^4)$, $z_{k-2}=O(h^4)$. Since the matrix of the system (70) is a diagonally dominant and its solution is estimated by the right-hand side, that has a $O(h^4)$ order. Thus we have

$$\omega_i=O(h^4),\ i=2(1)k-2.$$

Therefore from (69) it follows that (67) for $i=2(1)k-2$. The estimation (67) for $i=1$ and $i=k-1$ follows from (38) and (39), in which (52), (53) and (64) have been used for $i=0$ and $i=k$. Now we use (36) for $i=1$. We have

$$\alpha_0=6d_1-10\alpha_1-\alpha_2=y_0-\frac{h^2}{6}y_0^{''}+O(h^4),$$

where have been used (67) for $i=1$ and $i=2$. Analogously, from (36) for $i=k$, we have

$$\alpha_k=6d_{k-1}-10\alpha_{k-1}-\alpha_{k-2}=y_k-\frac{h^2}{6}y_k^{''}+O(h^4).$$

Thus (67) is proved for all $i=0(1)k$. Analogously, if we use (36) for $i=0$, and $i=k$ and (67) for $i=0,1$ and $i=k-1,k$, we obtain (66) and (68). This completes the proof. □

Remark 4.5 When $y(x) \in C^6[a,b]$, using (65), as well as equations (36), we obtain easily

$$\alpha_{-1} = y_0 - hy_0' + \frac{h^2}{3}y_0'' - \frac{19}{720}h^4y_0^{(4)} + \frac{h^5}{240}y_0^{(5)} + O(h^6), \tag{72}$$

$$\alpha_i = y_i - \frac{12}{6}y_i'' + \frac{11}{720}h^4y_i^{IV} + O(h^6), \quad i = 0(1)k, \tag{73}$$

$$\alpha_{k+1} = y_k + hy_k' + \frac{h^2}{3}y_k'' - \frac{19}{720}h^4y_k^{(4)} - \frac{h^5}{240}y_k^{(5)} + O(h^6). \tag{74}$$

Theorem 4.6. Let $S(x)$ be the integro cubic spline satisfying the conditions (3) and end conditions (56). Then

$$S_i - y_i = O(h^4), \quad i = 0(1)k, \tag{75}$$

$$m_i - y_i' = O(h^3), \quad i = 0(1)k, \tag{76}$$

$$M_i - y_i'' = O(h^2), \quad i = 0(1)k, \tag{77}$$

$$\frac{S_{i+0}''' + S_{i-0}'''}{2} - y_i''' = O(h), \; i = 1(1)k-1, \tag{78}$$

$$S_{i+0}''' - S_{i-0}''' = O(h) \quad i = 1(1)k-1. \tag{79}$$

Proof. By virtue of (16) and (67), we have

$$S_i = \frac{1}{6}(\alpha_{i-1} + 4\alpha_i + \alpha_{i+1}) = \frac{1}{6}(y_{i-1} + 4y_i + y_{i+1}) -$$

$$\frac{h^2}{36}(y_{i-1}'' + 4y_i'' + y_{i+1}'') + O(h^4) = y_i + O(h^4), \quad i = 1(1)k-1,$$

in which the Taylor expansion of function $y(x)$ is used. Analogously, by using formulas (66)-(68), one can obtain

$$S_0 = \frac{1}{6}(\alpha_{-1} + 4\alpha_0 + \alpha_1)$$

$$= \frac{1}{6}[y_0 - hy_0' + \frac{h^2}{3}y_0'' + 4(y_0 - \frac{h^2}{6}y_0'') + y_1 - \frac{h^2}{6}y_1''] + O(h^4) = y_0 + O(h^4),$$

and

$$S_k = \frac{1}{6}(\alpha_{k-1} + 4\alpha_k + \alpha_{k-1}) = y_k + O(h^4),$$

respectively. This means that (75) holds for all i, $i = 0(1)k$. From (18) and (67) it follows

$$M_i = \frac{\alpha_{i+1} - 2\alpha_i + \alpha_{i-1}}{h^2} = \frac{1}{h^2}[y_{i-1} - 2y_i + y_{i+1} -$$

$$\frac{h^2}{6}(y_{i-1}'' - 2y_i'' + y_{i+1}'')] + O(h^2) = y_i'' + O(h^2), \quad i = 1(1)k-1.$$

Using (65), (67) and (68) it is easy to verify the estimation (77) for $i = 0$ and $i = k$. Now we consider $\frac{S_{i+0}''' + S_{i-0}'''}{2}$. Using (77) we obtain

$$\frac{S_{i+0}''' + S_{i-0}'''}{2h} = \frac{M_{i+1} - M_{i-1}}{2h} = \frac{1}{2h}(y_{i+1}'' - y_{i-1}'') + O(h) =$$

$$\frac{1}{2h}(y_i^{''} + hy_i^{'''} - y_i^{''} + hy_i^{'''}) + O(h) = y_i^{'''} + O(h), \quad i = 1(1)k-1,$$

i.e., the estimation (78) is proven. Analogously, we have

$$S_{i+0}^{'''} - S_{i-0}^{'''} = \frac{M_{i+1} - 2M_i + M_{i-1}}{h}$$

$$= \frac{y_{i+1}^{''} - 2y_i^{''} + y_{i-1}^{''}}{h} + O(h) = O(h), \quad i = 1(1)k-1,$$

which completes the proof of Theorem 4.6. □

5. Numerical examples

In this section, we present results of the numerical experiment to illustrate the approximation properties of the integro cubic splines. Suppose that $y(x) \in C^4[0,1]$ and satisfies the end condition $M_1 = y'(x_1)$, we consider the following test functions

$$y_1(x) = x^4, \qquad y_2(x) = \cos(\pi x).$$

The results are shown in Table 1:

Table 1

	$\lvert S_j - y_{1,j}\rvert$			$\lvert m_j - y'_{1,j}\rvert$			$\lvert M_j - y''_{1,j}\rvert$		
x_j	$k=10$	$k=20$	$k=40$	$k=10$	$k=20$	$k=40$	$k=10$	$k=20$	$k=40$
0	1.62E-4	1.01E-5	6.31E-7	8.08E-3	1.01E-3	1.26E-04	2.18E-01	5.45E-2	1.36E-2
.1	2.00E-5	1.03E-7	1.30E-8	8.17E-4	1.03E-5	1.32E-08	5.55E-17	5.51E-3	1.25E-3
.2	1.65E-6	2.07E-7	1.30E-8	8.25E-5	1.05E-7	1.37E-12	2.20E-02	5.01E-3	1.25E-3
.3	3.50E-6	2.08E-7	1.30E-8	8.33E-6	1.07E-9	6.94E-16	1.98E-02	5.00E-3	1.25E-3
.4	3.32E-6	2.08E-7	1.30E-8	8.33E-7	1.10E-1	1.78E-15	2.00E-02	5.00E-3	1.25E-3
.5	3.34E-6	2.08E-7	1.30E-8	1.11E-6	2.55E-5	1.89E-14	2.00E-02	5.00E-3	1.25E-3
.6	3.32E-6	2.08E-7	1.30E-8	8.33E-7	1.10E-1	5.55E-15	2.00E-02	5.00E-3	1.25E-3
.7	3.50E-6	2.08E-7	1.30E-8	8.33E-6	1.07E-9	4.91E-14	1.98E-02	5.00E-3	1.25E-3
.8	1.65E-6	2.07E-7	1.30E-8	8.25E-5	1.05E-7	1.56E-12	2.20E-02	5.01E-3	1.25E-3
.9	2.00E-5	1.03E-7	1.30E-8	8.17E-4	1.03E-5	1.32E-08	3.93E-13	5.51E-3	1.25E-3
1	1.62E-4	1.01E-5	6.31E-7	8.08E-3	1.01E-3	1.26E-04	2.18E-01	5.45E-2	1.36E-2

	$\lvert S_j - y_{2,j}\rvert$			$\lvert m_j - y'_{2,j}\rvert$			$\lvert M_j - y''_{2,j}\rvert$		
x_j	$k=10$	$k=20$	$k=40$	$k=10$	$k=20$	$k=40$	$k=10$	$k=20$	$k=40$
0	6.21E-4	4.05E-5	2.55E-06	3.11E-2	4.05E-3	5.11E-4	8.44E-01	2.19E-1	5.51E-2
.1	7.70E-5	3.97E-7	5.02E-08	3.10E-3	4.38E-5	2.07E-7	2.31E-14	2.14E-2	4.83E-3
.2	5.33E-6	6.94E-7	4.30E-08	3.93E-4	5.11E-6	2.93E-7	7.44E-02	1.65E-2	4.11E-3
.3	8.41E-6	5.09E-7	3.13E-08	7.15E-5	6.46E-6	4.03E-7	4.66E-02	1.20E-2	2.99E-3
.4	4.74E-6	2.71E-7	1.65E-08	1.25E-4	7.59E-6	4.74E-7	2.59E-02	6.32E-3	1.57E-3
.5	4.80E-7	7.74E-9	1.22E-10	1.27E-4	7.98E-6	4.98E-7	5.76E-04	3.72E-5	2.34E-6
.6	3.78E-6	2.56E-7	1.62E-08	1.25E-4	7.59E-6	4.74E-7	2.47E-02	6.24E-3	1.57E-3
.7	9.37E-6	4.94E-7	3.10E-08	7.13E-5	6.46E-6	4.03E-7	4.77E-02	1.19E-2	2.98E-3
.8	4.34E-6	6.78E-7	4.27E-08	3.94E-4	5.11E-6	2.93E-7	7.32E-02	1.64E-2	4.10E-3
.9	7.83E-5	3.81E-7	5.00E-08	3.12E-3	4.38E-5	2.07E-7	7.68E-04	2.13E-2	4.83E-3
1	6.25E-4	4.05E-5	2.56E-06	3.13E-2	4.05E-3	5.11E-4	8.47E-01	2.19E-1	5.51E-2

As shown in Table 1, the approximation properties of integro cubic splines was confirmed by numerical experiments.

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